I. Kinematics

Consider an inertial frame $X_iY_iZ_i$ with origin $O_i$ and a body-fixed frame $X_bY_bZ_b$ with origin $O_b$. The rotation matrix $R^b_i$ which transforms vectors in body-fixed frame to the inertial frame can be parametrized in terms of three angles $\theta_x$, $\theta_y$, and $\theta_z$ as

$$R^b_i = R_{x,\theta_x}R_{y,\theta_y}R_{x,\theta_z}$$

where $R_{x,\theta_x}$ denotes the rotation matrix corresponding to a rotation about $X$-axis by the angle $\theta_x$, etc. Hence,

$$R^b_i = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_x & -s_x \\ 0 & s_x & c_x \end{bmatrix}$$

$$= \begin{bmatrix} c_y c_z & s_y c_z - c_x s_z & c_x s_y c_z + s_x s_z \\ c_x c_z & s_x s_y + c_y s_z & c_x s_y s_z - s_x c_z \\ -s_y c_z & s_x c_y & c_x c_y \end{bmatrix}$$

where $c_x = \cos(\theta_x)$, $s_x = \sin(\theta_x)$, etc.

The position and orientation of the body relative to the inertial frame can be represented by a $6 \times 1$ vector $p = [p^T_t, p^T_r]^T$ where $p_t = [x, y, z]^T$ represents the Cartesian coordinates of $O_b$ as measured in the inertial frame and $p_r = [\theta_x, \theta_y, \theta_z]^T$ specifies the rotation angles. The translational and angular velocities of the body relative to the inertial frame and expressed in the body-fixed frame are denoted as $v_t$ and $v_r$, respectively. Let $v = [v^T_t, v^T_r]^T$. Hence,

$$\dot{p}_t = R^b_i v_t = J_t(p_r)v_t \quad ; \quad J_t(p_r) = R^b_i.$$  

By definition, the angular velocity $v_r$ satisfies

$$\dot{R}^b_i = R^b_i S(v_r)$$

where $S(\omega)$ with $\omega = [\omega_x, \omega_y, \omega_z]^T$ denotes the skew-symmetric matrix

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$
Denoting by $e_1$, $e_2$, and $e_3$ the three principal unit vectors, i.e., $e_1 = [1, 0, 0]^T$, $e_2 = [0, 1, 0]^T$, and $e_3 = [0, 0, 1]^T$, we have

\[
\begin{align*}
\dot{R}_{x, \theta_x} &= \dot{\theta}_x R_{x, \theta_x} S(e_1) \\
\dot{R}_{y, \theta_y} &= \dot{\theta}_y R_{y, \theta_y} S(e_2) \\
\dot{R}_{z, \theta_z} &= \dot{\theta}_z R_{z, \theta_z} S(e_3). 
\end{align*}
\]  

(6)

Hence,

\[
\begin{align*}
\dot{b}'_i &= R_{z, \theta_z} R_{y, \theta_y} R_{x, \theta_x} \dot{\theta}_x S(e_1) + R_{z, \theta_z} R_{y, \theta_y} \dot{\theta}_y S(e_2) R_{x, \theta_x} + R_{z, \theta_z} \dot{\theta}_z S(e_3) R_{y, \theta_y} R_{x, \theta_x} \\
&= R_{z, \theta_z} R_{y, \theta_y} R_{x, \theta_x} [\dot{\theta}_x S(e_1) + R^T_{x, \theta_x} \dot{\theta}_y S(e_2) R_{x, \theta_x} + R^T_{x, \theta_x} R^T_{y, \theta_y} \dot{\theta}_z S(e_3) R_{y, \theta_y} R_{x, \theta_x}] \\
&= R^b_0 [\dot{\theta}_x S(e_1) + \dot{\theta}_y S(R^T_{x, \theta_x} e_2) + \dot{\theta}_z S(R^T_{x, \theta_x} R^T_{y, \theta_y} e_3)] \\
&= R^b_0 S(\dot{\theta}_x e_1 + \dot{\theta}_y R^T_{x, \theta_x} e_2 + \dot{\theta}_z R^T_{x, \theta_x} R^T_{y, \theta_y} e_3).
\end{align*}
\]  

(7)

Comparing with (4), $v_r$ is related to $p_r$ as

\[
v_r = \dot{\theta}_x e_1 + \dot{\theta}_y R^T_{x, \theta_x} e_2 + \dot{\theta}_z R^T_{x, \theta_x} R^T_{y, \theta_y} e_3
\]

\[
\implies v_r = [e_1; R^T_{x, \theta_x} e_2; R^T_{x, \theta_x} R^T_{y, \theta_y} e_3] \dot{p}_r
\]

\[
= \begin{bmatrix} 1 & 0 & -s_y \\ 0 & c_x & s_x c_y \\ 0 & -s_x & c_x c_y \end{bmatrix} \dot{p}_r.
\]  

(8)

Hence,

\[
\dot{p}_r = J_r(p_r) v_r
\]  

(9)

where

\[
J_r(p_r) = \begin{bmatrix} 1 & s_x t_y & c_x t_y \\ 0 & c_x & -s_x \\ 0 & s_x/c_y & c_x/c_y \end{bmatrix}
\]  

(10)

where $t_x = \tan(\theta_x)$ and $t_y = \tan(\theta_y)$. Using (3) and (9),

\[
\dot{p} = J(p_r) v
\]  

(11)

where\(^2\)

\[
\begin{bmatrix} J_r(p_r) & 0_{3 \times 3} \\ 0_{3 \times 3} & J_r(p_r) \end{bmatrix}
\]

\[^2\] $I_{n \times n}$ and $0_{n \times n}$ denote the $n \times n$ identity matrix and zero matrix, respectively.
\[
\begin{bmatrix}
  c_y c_z & s_x s_y c_z - c_x s_z & c_x s_y c_z + s_x s_z & 0 & 0 & 0 \\
  c_y s_z & s_x s_y s_z + c_x c_z & c_x s_y s_z - s_x c_z & 0 & 0 & 0 \\
  -s_y & s_x c_y & c_x c_y & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & s_x t_y & c_x t_y \\
  0 & 0 & 0 & 0 & c_x & -s_x \\
  0 & 0 & 0 & 0 & s_x/c_y & c_x/c_y
\end{bmatrix}
\]

(12)

II. Dynamics

The translational dynamics of a rigid body are obtained from the observation that the mass of the body times the acceleration of the center of mass is equal to the net force acting on the body. Denote by \( G \) the center of mass of the rigid body. Let \( p_{G,i} \) and \( p_{G,b} \) denote the coordinates of \( G \) relative to the inertial and body-fixed frames, respectively. Then,

\[
p_{G,i} = p_t + R_{i}^{b} p_{G,b}.\tag{13}
\]

Differentiating (13) and using (3) and (4) yields

\[
\dot{p}_{G,i} = R_{i}^{b} v_t + R_{i}^{b} S(v_r)p_{G,b} \tag{14}
\]

\[
\Rightarrow \ddot{p}_{G,i} = R_{i}^{b} [\dot{v}_t + S(\dot{v}_r)p_{G,b}] + R_{i}^{b} S(v_r)[v_t + S(v_r)p_{G,b}] . \tag{15}
\]

Denoting the mass of the body by \( m \) and the net force acting on the body expressed in the body-fixed frame by \( F \), the translational dynamics of the rigid body are given by

\[
m \ddot{p}_{G,i} = R_{i}^{b} F. \tag{16}
\]

Using (15) and (16), the translational dynamics can be written in terms of the translational and angular velocities expressed in the body-fixed frame as

\[
m[\ddot{v}_t + S(\dot{v}_r)p_{G,b} + S(v_r)v_t + S(v_r)S(v_r)p_{G,b}] = F. \tag{17}
\]

Using the property of skew-symmetric matrices that \( S(a)b = a \times b = -b \times a = -S(b)a \), (17) can be rewritten as

\[
m[\ddot{v}_t - S(p_{G,b})\dot{v}_r + S(v_r)v_t - S(v_r)S(p_{G,b})v_r] = F. \tag{18}
\]

To facilitate the derivation of the rotational dynamics, consider a decomposition of the rigid body into infinitesimal volume elements \( dV \) with local densities \( \rho_{dV} \). Let \( p_{dV,i} \) and \( p_{dV,b} \) denote the coordinates of the infinitesimal element \( dV \) relative to the inertial and body-fixed frames,

\[a \times b \text{ with } a \text{ and } b \text{ being vectors denotes the vector cross-product of } a \text{ and } b.\]
respectively, and let $v_{dV,i}$ denote the velocity of $dV$ expressed in the inertial frame. Note that each $p_{dV,b}$ is constant by the definition of a rigid body. The net torque about the point $O_b$ acting on the rigid body is given in the inertial frame by

$$\tau_i = \int [(p_{dV,i} - p_t) \times \dot{v}_{dV,i}] \rho_{dV} dV$$

(19)

where the integral is carried out over all the volume elements composing the rigid body. Hence, the net torque expressed in the body-fixed frame is

$$\tau = R_i^b \tau_i = R_i^b \int [(p_{dV,i} - p_t) \times \dot{v}_{dV,i}] \rho_{dV} dV.$$  

(20)

Using the relations

$$p_{dV,i} = p_t + R_i^b p_{dV,b}$$

(21)

$$v_{dV,i} = R_i^b [\dot{v}_t + S(v_r)p_{dV,b}]$$

(22)

$$\dot{v}_{dV,i} = R_i^b [\dot{v}_t + S(\dot{v}_r)p_{dV,b} + (\dot{v}_r) v_t + S(v_r)S(v_r)p_{dV,b}]$$

(23)

$$(p_{dV,i} - p_t) \times \dot{v}_{dV,i} = R_i^b \{p_{dV,b} \times [\dot{v}_t + S(\dot{v}_r)p_{dV,b} + (\dot{v}_r) v_t + S(v_r)S(v_r)p_{dV,b}]}.$$  

(24)

(20) can be expanded as

$$\tau = \int \{p_{dV,b} \times [\dot{v}_t + S(\dot{v}_r)p_{dV,b} + (\dot{v}_r) v_t + S(v_r)S(v_r)p_{dV,b}]} \rho_{dV} dV.$$  

(25)

The mass $m$ and the moment of inertia $I_b$ (about the point $O_b$) satisfy, by definition,

$$m p_{G,b} = \int p_{dV,b} \rho_{dV} dV$$

(26)

$$I_b \dot{v}_r = \int \{p_{dV,b} \times [S(\dot{v}_r)p_{dV,b}]\} \rho_{dV} dV$$

(27)

$$I_b = - \int S(p_{dV,b})S(p_{dV,b}) \rho_{dV} dV.$$  

(28)

Note that $I_b$ is a constant symmetric positive-definite matrix. Using the vector triple product identity given by

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$$

(29)

valid for any vectors $a$, $b$, and $c$, we obtain

$$p_{dV,b} \times [S(v_r)S(v_r)p_{dV,b}] = S(p_{dV,b})S(v_r)[v_r \times p_{dV,b}] = -S(v_r)S(v_r)p_{dV,b} - S(v_r \times p_{dV,b})v_r = -S(v_r)p_{dV,b} - S(v_r \times p_{dV,b})(p_{dV,b} \times v_r) = S(v_r)S(p_{dV,b})(v_r \times p_{dV,b})$$

valid for any vectors $a$, $b$, and $c$, we obtain
\[ S(v_r)S(p_{AV;b})S(p_{AV;b})v_r. \] (30)

Hence,

\[
\int \{ p_{AV;b} \times [ S(v_r)S(v_r)p_{AV;b} ] \} p_{AV}dV \quad = \quad S(v_r)I_b v_r \quad = \quad -S(I_b v_r) v_r. \] (31)

Using (26), (27), (28), and (31), (25) reduces to

\[
\tau \quad = \quad mS(p_{G;b})\dot{v}_t + I_b \dot{v}_r + mS(p_{G;b})S(v_t)v_t - S(I_b v_r)v_r. \] (32)

The rigid body kinematics given by (11) and the translational and rotation dynamics given by (18) and (32) can be rewritten as

\[
\dot{p} \quad = \quad J(p)v \] (33)

\[
M_{RB} \ddot{v} + C_{RB}(v)v \quad = \quad F \] (34)

where

\[
M_{RB} = \begin{bmatrix}
mI_{3\times3} & -mS(p_{G;b}) \\
mS(p_{G;b}) & I_b
\end{bmatrix} \] (35)

\[
C_{RB}(v) = \begin{bmatrix}
mS(v_r) & -mS(v_r)S(p_{G;b}) \\
mS(p_{G;b})S(v_r) & -S(I_b v_r)
\end{bmatrix} \] (36)

and \( F \) is the 6 \times 1 generalized force vector given by \( F = [F^T, \tau^T]^T \). Note that \( M_{RB} \) is a constant symmetric positive-definite matrix while \( C_{RB}(v) \) is a skew-symmetric matrix that depends on the angular velocity of the rigid body. Introducing the notations

\[
p_{G;b} = [p_{G;b;x}, p_{G;b;y}, p_{G;b;z}]^T \] (37)

\[
v_t = [v_{t,x}, v_{t,y}, v_{t,z}]^T \] (38)

\[
v_r = [v_{r,x}, v_{r,y}, v_{r,z}]^T \] (39)

\[
F = [F_x, F_y, F_z]^T \] (40)

\[
\tau = [\tau_x, \tau_y, \tau_z]^T \] (41)

\[
I_b = \begin{bmatrix}
I_{b,xx} & I_{b,xy} & I_{b,xz} \\
I_{b,xy} & I_{b,yy} & I_{b,yz} \\
I_{b,xz} & I_{b,yz} & I_{b,zz}
\end{bmatrix}, \] (42)
\(M_{RB}\) and \(C_{RB}(v)\) are obtained as

\[
M_{RB} = \begin{bmatrix}
m & 0 & 0 & 0 & mp_{G,b,z} & -m_{G,b,y} \\
0 & m & 0 & -mp_{G,b,z} & 0 & mp_{G,b,x} \\
0 & 0 & m & mp_{G,b,y} & -mp_{G,b,x} & 0 \\
0 & -mp_{G,b,z} & m_{G,b,y} & I_{b,xx} & I_{b,xy} & I_{b,xz} \\
mp_{G,b,z} & 0 & -mp_{G,b,x} & I_{b,yy} & I_{b,yz} & I_{b,zz} \\
-m_{G,b,y} & mp_{G,b,x} & 0 & I_{b,xz} & I_{b,yz} & I_{b,zz}
\end{bmatrix}
\]

(43)

\[
C_{RB}(v) = \begin{bmatrix}
C_{11}(v) & C_{12}(v) \\
C_{21}(v) & C_{22}(v)
\end{bmatrix}
\]

(44)

where

\[
C_{11}(v) = \begin{bmatrix}
0 & -mv_{r,z} & mv_{r,y} \\
-mv_{r,z} & 0 & -mv_{r,x} \\
-mv_{r,y} & mv_{r,x} & 0
\end{bmatrix}
\]

(45)

\[
C_{12}(v) = \begin{bmatrix}
v_{r,z}p_{G,b,z} + v_{r,y}p_{G,b,y} & -v_{r,y}p_{G,b,x} & -v_{r,z}p_{G,b,y} \\
-v_{r,x}p_{G,b,z} & v_{r,z}p_{G,b,x} + v_{r,x}p_{G,b,z} & -v_{r,x}p_{G,b,y} \\
-v_{r,y}p_{G,b,z} & -v_{r,x}p_{G,b,y} & v_{r,y}p_{G,b,x} + v_{r,x}p_{G,b,z}
\end{bmatrix}
\]

(46)

\[
C_{21}(v) = \begin{bmatrix}
v_{r,z}p_{G,b,z} - v_{r,y}p_{G,b,y} & v_{r,x}p_{G,b,y} & v_{r,x}p_{G,b,z} \\
v_{r,y}p_{G,b,z} & v_{r,z}p_{G,b,x} - v_{r,x}p_{G,b,z} & v_{r,y}p_{G,b,z} \\
v_{r,x}p_{G,b,z} & -v_{r,y}p_{G,b,x} & v_{r,x}p_{G,b,z} - v_{r,y}p_{G,b,z}
\end{bmatrix}
\]

(47)

\[
C_{22}(v) = \begin{bmatrix}
0 & (I_{b,xx}v_{r,x} + I_{b,yy}v_{r,y} + I_{b,xz}v_{r,z}) & -(I_{b,xy}v_{r,x} + I_{b,yy}v_{r,y} + I_{b,yz}v_{r,z}) \\
(I_{b,xx}v_{r,x} + I_{b,yy}v_{r,y} + I_{b,xz}v_{r,z}) & 0 & (I_{b,xy}v_{r,x} + I_{b,yy}v_{r,y} + I_{b,xz}v_{r,z}) \\
-(I_{b,xy}v_{r,x} + I_{b,yy}v_{r,y} + I_{b,yz}v_{r,z}) & (I_{b,xy}v_{r,x} + I_{b,yy}v_{r,y} + I_{b,xz}v_{r,z}) & 0
\end{bmatrix}
\]

(48)