Euler-Lagrange formulation for dynamics of an n-link manipulator

In the Euler-Lagrange dynamics formulation, the dynamics of an n-link manipulator are written as:

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \quad i = 1, \ldots, n
\]

(1)

where the Lagrangian \( \mathcal{L} \) is defined as \( \mathcal{L} = K - P \) with \( K \) being the kinetic energy of the system and \( P \) being the potential energy of the system. \( \tau_i \) is the force/torque corresponding to the \( i^{th} \) joint of the manipulator.

Given an \( n \)-link manipulator, the kinetic energy of the manipulator can be written as:

\[
K = \frac{1}{2} \dot{q}^T D(q) \dot{q}
\]

(2)

with \( D(q) \) defined as

\[
D(q) = \sum_{i=1}^{n} \left\{ m_i J^T_{c,i} J_{c,i} + J^T_{c,i} R^0_i R^0_i J_{c,i} \right\}.
\]

(3)

Here, \( m_i \) denotes the mass of the \( i^{th} \) link, \( I_i \) denotes the inertia matrix in the link-fixed frame with its origin at the center of mass of the link, \( J_{c,i} \) denotes the velocity Jacobian for the center of mass of link \( i \), and \( \omega_{c,i} \) denotes the angular velocity (written relative to frame 0) of link \( i \) is written as \( v_{c,i}^{(0)} = J_{c,i}(q) \dot{q} \) and the angular velocity (written relative to frame 0) of link \( i \) is written as \( \omega_{c,i}^{(0)} = J_{c,i}(q) \dot{q} \).

Note that \( D(q) \) as defined in equation (3) is a symmetric matrix.

**Inertia matrix:** Note that since \( I_i \) is the inertia matrix written relative to the link-fixed frame, \( R^0_i I_i (R^0_i)^T \) is the inertia matrix written relative to an inertial frame (with the origin of the frame at the center of mass of the link). The inertia matrix is typically a constant matrix when written in the link-fixed frame.

The inertia matrix \( I_i \) is a \( 3 \times 3 \) symmetric matrix whose elements can be found by a volume integration, i.e.,

\[
I_i = \begin{bmatrix}
I_{xx} & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} & I_{yz} \\
I_{zx} & I_{zy} & I_{zz}
\end{bmatrix}
\]

(4)

where \( I_{xx} = \int \int \int (y^2 + z^2) \rho(x, y, z) dx \, dy \, dz \), \( I_{yy} = \int \int \int (x^2 + z^2) \rho(x, y, z) dx \, dy \, dz \), \( I_{zz} = \int \int \int (x^2 + y^2) \rho(x, y, z) dx \, dy \, dz \), \( I_{xy} = I_{yx} = -\int \int \int xy \rho(x, y, z) dx \, dy \, dz \), \( I_{xz} = I_{zx} = -\int \int \int xz \rho(x, y, z) dx \, dy \, dz \), \( I_{yz} = I_{zy} = -\int \int \int yz \rho(x, y, z) dx \, dy \, dz \).

The potential energy of the \( n \)-link manipulator can be written as

\[
P = \sum_{i=1}^{n} m_i g^T r_{ci}
\]

(5)

where \( g \) is the acceleration due to gravity (written relative to frame 0) and \( r_{ci} \) is the position of the center of mass of link \( i \) (again, written relative to frame 0).

If the kinetic energy and potential energy functions that were found as in equations (2) and (5) are algebraically simple, then it is easy to simply substitute \( \mathcal{L} = K - P \) into the Euler-Lagrange equation (1) to find the dynamics equations. Alternatively, a more formal procedure is to use the Christoffel symbols defined below.

From the matrix \( D(q) \) that was found in equation (3), the Christoffel symbols \( c_{ijk} \) are found as:

\[
c_{ijk} = \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\}
\]

(6)

where expressions such as \( d_{ij} \) denote the \((i,j)^{th}\) element, etc., of the matrix \( D(q) \). The Christoffel symbols need to be found for all \( i, j, k \) in \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}, k \in \{1, \ldots, n\} \). In writing the Christoffel symbols, we can use the property that \( c_{ijk} = c_{jik} \) to reduce the number of Christoffel symbols that need to be explicitly calculated by around a half.

From the potential energy (5), define the functions

\[
g_k(q) = \frac{\partial P}{\partial q_k} \quad k = 1, \ldots, n.
\]

(7)

The Euler-Lagrange dynamics equations can be written as:

\[
\sum_{i=1}^{n} d_{kj}(q) \ddot{q}_j + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijk}(q) \dot{q}_i \dot{q}_j + g_k(q) = \tau_k \quad k = 1, \ldots, n.
\]

(8)
Then, the Euler-Lagrange dynamics equations from (8) can be written in a matrix form as
\[ D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \]
where \( g(q) = [g_1(q), \ldots, g_n(q)]^T \) and \( \tau = [\tau_1, \ldots, \tau_n]^T \).

Some properties of the \( D \) and \( C \) matrices:

- The \( D(q) \) matrix is symmetric and positive-definite.
- The matrix \( N(q, \dot{q}) \) defined as \( N(q, \dot{q}) = \dot{D}(q) - 2C(q, \dot{q}) \) is skew symmetric, i.e., \([N(q, \dot{q})]^T = -N(q, \dot{q})\).

Derivation of equation (8): Denoting the \((i, j)\)th element of the matrix \( D(q) \) by \( d_{ij} \), the kinetic energy of the manipulator is seen from equation (2) to be of the form
\[ K = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j. \]
The potential energy \( P \) depends only on \( q \) and does not depend on \( \dot{q} \). Hence, we see that for any \( k \) in \( 1, \ldots, n \):
\[ \frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j. \]
Hence,
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j + \sum_{j=1}^{n} \left\{ \frac{d}{dt} d_{kj}(q) \right\} \dot{q}_j = \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j + \sum_{j=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j. \]
Also, note that
\[ \frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial d_{kj}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial P}{\partial q_k}. \]
Hence, the Euler-Lagrange equation (1) can be written as:
\[ \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k. \]
By interchanging the dummy variables of summation, we can write
\[ \sum_{j=1}^{n} \sum_{i=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \left\{ \frac{\partial d_{ki}}{\partial q_j} \dot{q}_i \dot{q}_j. \right\} \dot{q}_i \dot{q}_j. \]
Hence,
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} \dot{q}_i \dot{q}_j. \right\} \dot{q}_i \dot{q}_j. \]
Therefore, from (16), we get
\[ \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} \dot{q}_i \dot{q}_j - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k. \]
Hence, from the definition of the Christoffel symbols from (6), we get the dynamics equations shown in equation (8).