8.2
a) – This system has the matrices
\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & -2
\end{bmatrix} \quad B = \begin{bmatrix} 1 \\
0 \\
1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.
\]

For observability we can test,
\[
(\Gamma_3^*)^T = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix} \implies \text{rank}(\Gamma_3^*)^T = 2 < 3
\]
since $(\Gamma_3^*)^T$ has only two linearly independent columns. Therefore the system is not observable.

b) – We have
\[
A = \begin{bmatrix}
-4 & 3/2 & 3 \\
-1 & 0 & 1 \\
-5 & 3 & 3
\end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\
0 & 1 \\
2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}
\]

For observability form,
\[
\Gamma_3^* = \begin{bmatrix} 1 & 1 & -5/2 \\ 0 & -3/2 & 3/2 \\ -1 & 0 & 3/2 \end{bmatrix}.
\]

When we add the second column to the third column we get
\[
\hat{\Gamma}_3^* = \begin{bmatrix} 1 & 1 & -3/2 \\ 0 & -3/2 & 0 \\ -1 & 0 & 3/2 \end{bmatrix} \implies \text{rank}(\hat{\Gamma}_3^*) = 2 < 3
\]
since the third column of $\hat{\Gamma}_3^*$ is just a multiple of the first column. Therefore the system is not observable.

c) – For observability form,
\[
\Gamma_n^* = I_n \implies \text{rank}(\Gamma_n^*) = n.
\]

Therefore the system is observable.

8.3 – For this system
\[
A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.
\]
b) – We find that
\[
\Gamma_3^* = \begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T \end{bmatrix} = \begin{bmatrix}
1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & -3 & 4 \\
1 & -1 & 2 & -2 & 1 & -4 \\
\end{bmatrix}.
\]
We note that the first 3 columns of \(\Gamma_3^*\) are linearly independent because their determinant is not zero. Therefore the rank of \(\Gamma_3^*\) is three, so the system is observable.

8.4 – For this system
\[
A = \begin{bmatrix} -2 & -3 & 4 \\
0 & -2 & 2 \\
-2 & -1 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\
0 \\
1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix}.
\]
a) – Therefore
\[
\Gamma_3 = \begin{bmatrix} B & AB & A^2 B \end{bmatrix} = \begin{bmatrix}
1 & 1 & 2 & -5 & -10 & 4 \\
0 & 1 & 2 & -4 & -2 \\
1 & 0 & 0 & -3 & -6 & 6 \\
\end{bmatrix}.
\]
We see that the first row of \(\Gamma_3\) is just the sum of the second and third rows. Therefore the rank of \(\Gamma_3\) is not equal to three, so the system is not controllable.

We find that
\[
(\Gamma_3^*)^T = \begin{bmatrix} C \\
CA \\
CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\
-2 & -3 & 4 \\
-4 & 8 & -6 \end{bmatrix}.
\]
We note that the determinant of \((\Gamma_3^*)^T = -14\) is not zero. Therefore the rank of \((\Gamma_3^*)^T\) is three, so the system is observable.

b) – The controllability and observability properties are the same when these are the system matrices of a discrete system.

8.5 – For this system
\[
A = \begin{bmatrix} -2 & 1 & 0 \\
0 & -2 & 2 \\
-2 & -1 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & -1 \\
0 & 1 \\
1 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 \\
0 \\
1 \end{bmatrix}.
\]
b) – We find that
\[
\Gamma_3^* = \begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T \end{bmatrix} = \begin{bmatrix}
1 & -4 & 4 \\
0 & 0 & -6 \\
1 & 2 & 4 \end{bmatrix}.
\]
We note that the determinant of \(\Gamma_3^* = 36\) which is not zero. Therefore the rank of \(\Gamma_3^*\) is three, and the system is observable.
8.6 – For this system

\[
A = \begin{bmatrix}
\frac{3}{4} & -\frac{1}{4} & -\frac{3}{2} \\
-\frac{1}{4} & \frac{3}{4} & \frac{5}{2} \\
0 & 0 & 1
\end{bmatrix}; \quad B = \begin{bmatrix} 0 & 3 \\ 2 & -2 \\ 0 & \frac{1}{2} \end{bmatrix}; \quad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}.
\]

To test for controllability, form

\[
\Gamma_3 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 3 & -\frac{1}{2} & 2 & -\frac{3}{4} & 1 \\ 2 & -2 & \frac{3}{2} & -1 & \frac{5}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.
\]

If we take the first three columns of \( \Gamma_3 \) we get

\[
\hat{\Gamma} = \begin{bmatrix} 0 & 3 & -\frac{1}{2} \\ 2 & -2 & \frac{3}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.
\]

Since \( \det \hat{\Gamma} = \frac{1}{2} \neq 0 \), the rank of \( \Gamma_3 \) is 3 and the system is controllable.

To test for observability form

\[
\Gamma^*_3 = \begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{8} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{8} & 0 \\ -1 & 2 & -\frac{1}{2} & 2 & -\frac{1}{4} & 2 \end{bmatrix}.
\]

We note that the first and second rows of \( \Gamma^*_3 \) are the same. Therefore \( (\Gamma^*_3)^T \) has only two linearly independent rows so the rank of \( \Gamma^*_3 \) is two, and the system is not observable.

8.7 b) – To test observability, form

\[
w(\tau) = C(\tau)\phi(\tau, t_0)\mu = \mu_1 e^{-(2\tau-t_0)} + \mu_2 \left( e^{-(2\tau-t_0)} - \frac{\tau + \frac{1}{2}}{t_0 + \frac{1}{4}} e^{-\tau} \right).
\]

The only constant vector \( \mu \) that makes this identically zero is \( \mu = 0 \), therefore by condition 5 the system is observable.

8.8

The system state equation is

\[
\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{bmatrix} u
\]

The state transition matrix for the system is

\[
\phi(t-t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t-t_0 & 1 \end{bmatrix}
\]
a) – When the system is time-invariant, the $\Gamma_3$ matrix is

$$\Gamma_3 = \begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & b_2 & 0 \end{bmatrix} \implies \text{rank}(\Gamma_3) < 3$$

so the system is never controllable.

b) – We will first try the sufficient condition. Thus form the matrices

$$M_0(t) = \begin{bmatrix} b_1 \\ t \\ b_3 \end{bmatrix}; \quad M_1(t) = \dot{M}_0(t) - AM_0(t) = \begin{bmatrix} 0 \\ 1 \\ -t \end{bmatrix}; \quad M_2(t) = \dot{M}_1(t) - AM_1(t) = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$ 

Therefore

$$\Gamma_3(t) = \begin{bmatrix} b_1 & 0 & 0 \\ t & 1 & 0 \\ b_3 & -t & -2 \end{bmatrix}$$

Since $\det \Gamma_3(t) = -2b_1$, when $b_1 \neq 0$, the matrix $\Gamma_3(t)$ is full rank, and therefore the system is controllable. When $b_1 = 0$, we can see from the system diagram that $x_1$ is not affected by the control, and therefore the system is not controllable.

c) – In this case,

$$z(\tau) = B^T(\tau)\phi^T(t_0, \tau)\mu = \mu_1 b_1 + \mu_2 b_2 + \mu_3[(t_0 - \tau)b_2 + \tau].$$

The vector $\lambda = [ -b_2 \quad b_1 \quad 0 ]$ makes $z(\tau) = 0 \ \forall \tau$. Therefore the system is not controllable by condition 2.

d) – In this case,

$$z(\tau) = B^T(\tau)\phi^T(t_0, \tau)\mu = \mu_1 \tau + \mu_2 2\tau + \mu_3[(t_0 - \tau)2\tau + b_3].$$

The vector $\mu^T = [ -2 \quad 1 \quad 0 ]$ makes $z(\tau) = 0 \ \forall \tau$. Therefore the system is not controllable by condition 2.

e) – In this case, for the constant vector $\mu^T = [ \mu_1 \quad \mu_2 \quad \mu_3 ]$

$$z(\tau) = B^T(\tau)\phi^T(t_0, \tau)\mu = \mu_1 b_1(\tau) + \mu_2 b_2(\tau) + \mu_3[(t_0 - \tau)b_2(\tau) + b_3].$$

Clearly, since $b_2(\tau)$ is not a constant multiple of $b_1(\tau)$, $z(\tau)$ can not be identically equal to zero for any interval larger than 1. Therefore the system is controllable by condition 2.