1. (a) \(e^{2t}u(t)\): We know that the Laplace transform of \(e^{at}u(t)\) is \(\frac{1}{s-a}\) with the ROC \(\text{Re}(s) > \text{Re}(a)\). Hence, the Laplace transform of \(e^{2t}u(t)\) is \(\frac{1}{s-2}\) with the ROC \(\text{Re}(s) > 2\).

(b) \(\sin(t)\cos(t)u(t)\): The given signal can be simplified to \(\frac{1}{2}\sin(2t)u(t) = \frac{1}{4j}[e^{2jt}-e^{-2jt}]u(t)\). Hence, the Laplace transform of the given signal is

\[
\frac{1}{4j} \left\{ \frac{1}{s-2j} - \frac{1}{s+2j} \right\} = \frac{1}{s^2+4}. \tag{1}
\]

and the ROC is \(\text{Re}(s) > 0\).

2. Denoting \(x(t) = \cos(\omega t)u(t)\), we have \(X(s) = \frac{s}{s^2+\omega^2}\). The Laplace transform of \(e^{at}x(t) = e^{at}\cos(\omega t)u(t)\) is therefore \(\frac{s-a}{(s-a)^2+\omega^2}\).

3. For both the systems given, the method of one-sided Laplace transforms is probably easier than the method of guessing exponentials especially if your calculator can compute partial fraction expansions. Solutions using both the method of one-sided Laplace transforms and the method of guessing exponentials are given below.

(a) Taking one-sided Laplace transforms of both sides, we have

\[
s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = X(s). \tag{2}
\]

Therefore,

\[
Y(s) = \frac{X(s) + sy(0) + \dot{y}(0)}{s^2+1} = \frac{1}{s^2+1} + \frac{s}{s^2+1} = \frac{1 + s(s^2+1)}{(s^2+1)^2} \tag{3}
\]

where we have used the fact that the one-sided Laplace transform of \(\sin(t)\) is \(\frac{1}{s^2+1}\). Decomposing into partial fractions, we get

\[
Y(s) = \frac{0.5 + 0.25j}{s+j} + \frac{-0.25}{(s+j)^2} + \frac{0.5 - 0.25j}{s-j} + \frac{-0.25}{(s-j)^2}. \tag{4}
\]

Hence, for all \(t \geq 0\)

\[
y(t) = (0.5 + 0.25j)e^{jt} + (-0.25)te^{-jt} + (0.5 - 0.25j)e^{jt} + (-0.25)te^{jt} = 2\text{Re}((0.5 + 0.25j)e^{jt}) + 2(-0.25)t \cos(t) = \cos(t) + 0.5\sin(t) - 0.5t \cos(t). \tag{5}
\]

Alternatively, since the poles of the system are \(j\) and \(-j\) and the poles of the input signal are also \(j\) and \(-j\), we can guess that for all \(t \geq 0\), \(y(t)\) is a linear combination of \(e^{jt}, te^{jt}, e^{-jt},\) and \(te^{-jt}\). Since \(e^{jt} = \cos(t) + j\sin(t)\), etc., \(y(t)\) can be written as a linear combination of \(\sin(t)\), \(\cos(t)\), \(t \sin(t)\), and \(t \cos(t)\):

\[
y(t) = c_1 \sin(t) + c_2 \cos(t) + c_3 t \sin(t) + c_4 t \cos(t) \text{ for all } t \geq 0. \tag{6}
\]

Hence,

\[
\dot{y}(t) = (c_1 + c_4) \cos(t) + (c_3 - c_2) \sin(t) + c_3 t \cos(t) - c_4 t \sin(t) \tag{7}
\]

Using the initial conditions \(y(0) = 1, \dot{y}(0) = 0\), we get

\[
c_2 = 1 \\
c_1 + c_4 = 0. \tag{8}
\]

Also, using the equation \(\dot{y}(t) + y(t) = \sin(t)\), we get

\[
-2c_4 = 1 \\
2c_3 = 0. \tag{9}
\]

Hence, \(c_1 = 0.5, c_2 = 1, c_3 = 0,\) and \(c_4 = -0.5\) implying that \(y(t) = 0.5 \sin(t) + \cos(t) - 0.5t \cos(t)\) for all \(t \geq 0\).
(b) Taking one-sided Laplace transforms of both sides, we have

\[ s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) + Y(s) = X(s). \quad (10) \]

Therefore,

\[
Y(s) = \frac{X(s) + sy(0) + \dot{y}(0) + 2y(0)}{s^2 + 2s + 1} = \frac{1}{s^2 + 2s + 1} + \frac{1}{s + 3}
\]

\[
= \frac{1}{(s^2 + 2s + 1)(s + 2)}.
\quad (11)
\]

Decomposing into partial fractions, we get

\[
Y(s) = -\frac{1}{s + 1} + \frac{2}{(s + 1)^2} + \frac{1}{s + 2}.
\quad (12)
\]

Hence, for all \( t \geq 0 \),

\[
y(t) = -e^{-t} + 2te^{-t} + e^{-2t}.
\quad (13)
\]

Alternatively, since the system has a twice-repeated pole at \(-1\) and the input signal contributes a pole at \(-2\), we can guess that for all \( t \geq 0 \), \( y(t) \) is of the form

\[
y(t) = c_1e^{-t} + c_2te^{-t} + c_3e^{-2t}.
\quad (14)
\]

Hence,

\[
\dot{y}(t) = (c_2 - c_1)e^{-t} - c_2te^{-t} - 2c_3e^{-2t}
\]

\[
\ddot{y}(t) = (-2c_2 + c_1)e^{-t} + c_2te^{-t} + 4c_3e^{-2t}.
\quad (15)
\]

Using the initial conditions \( y(0) = 0, \dot{y}(0) = 1 \), we get

\[
c_1 + c_3 = 0
\]

\[
c_2 - c_1 - 2c_3 = 1.
\quad (16)
\]

Also, using the equation \( \ddot{y}(t) + 2\dot{y}(t) + y(t) = e^{-2t} \), we get

\[
c_3 = 1.
\quad (17)
\]

Hence, \( c_1 = -1, c_2 = 2 \), and \( c_3 = 1 \). Therefore, \( y(t) = -e^{-t} + 2te^{-t} + e^{-2t} \) for all \( t \geq 0 \).