EE3054 - Homework 5 - Solutions

1. We know the general result that the $z$-transform of $\alpha^n u[n]$ is $\frac{1}{1-\alpha z^{-1}}$ with ROC $|\alpha| < |z| < \infty$ and the $z$-transform of $-\alpha^n u[-n-1]$ is $\frac{1}{1-\alpha z^{-1}}$ with ROC $0 \leq |z| < |\alpha|$. Using this result, the inverse $z$-transforms of the given functions can be found as follows:

(a) $\frac{z}{z - 2}$ with ROC $2 < |z| < \infty$: Noting that $\frac{z}{z - 2} = \frac{1}{1-\frac{2}{z}}$, we obtain the inverse $z$-transform of $\frac{z}{z - 2}$ to be $2^n u[n]$. Observe that the signal $2^n u[n]$ is causal and hence the ROC of its $z$-transform is the exterior of a circle.

(b) $\frac{z}{z - 2}$ with ROC $0 < |z| < 2$: Note that $\frac{z}{z - 2} = \frac{1}{1-\frac{2}{z}}$. The ROC is given to be the interior of the circle with radius 2. Hence, the inverse $z$-transform is the anti-causal signal $-2^n u[-n-1]$.

(c) $\frac{3}{1 - 2.5z^{-1} + z^{-2}}$ with ROC $0.5 < |z| < 2$: The given function $\frac{3}{1 - 2.5z^{-1} + z^{-2}}$ has two poles since the denominator is a second order polynomial. The locations of the two poles can be found by solving the quadratic equation

$$1 - 2.5z^{-1} + z^{-2} = 0 \quad (1)$$

which is equivalent to the quadratic equation

$$z^2 - 2.5z + 1 = 0. \quad (2)$$

The roots of the quadratic equation (2) are located at 2 and 0.5. Hence, the given $z$-transform function can be decomposed into partial fractions as follows:

$$\frac{3}{1 - 2.5z^{-1} + z^{-2}} = \frac{A_1}{1 - 2z^{-1}} + \frac{A_2}{1 - 0.5z^{-1}} \quad (3)$$

where $A_1$ and $A_2$ are constants. To find $A_1$ and $A_2$, multiply both sides of (3) by $1 - 2.5z^{-1} + z^{-2}$ to obtain

$$3 = A_1(1 - 0.5z^{-1}) + A_2(1 - 2z^{-1}). \quad (4)$$

The equation (4) yields the following two equations in terms of $A_1$ and $A_2$:

$$3 = A_1 + A_2 \quad (5)$$
$$0 = -0.5A_1 - 2A_2. \quad (6)$$
Hence, \( A_1 = 4 \) and \( A_2 = -1 \). Therefore,

\[
\frac{3}{1 - 2.5z^{-1} + z^{-2}} = \frac{4}{1 - 2z^{-1}} - \frac{1}{1 - 0.5z^{-1}}. \tag{7}
\]

The ROC is given to be \( 0.5 < |z| < 2 \). Note that the pole of the first component in (7) is 2 while the pole of the second component is 0.5. Hence, the given ROC is in the interior of the circle through the pole of the first component and is in the exterior of the circle through the pole of the second component. Therefore, the first component in (7) yields an anti-causal term in the inverse \( z \)-transform while the second component in (7) yields a causal term in the inverse \( z \)-transform. Hence, the inverse \( z \)-transform of

\[
\frac{3}{1 - 2.5z^{-1} + z^{-2}} \quad \text{with the ROC } 0.5 < |z| < 2
\]

is

\[
4(-2^n u[(-n - 1)]) - (0.5^n)u[n]. \tag{8}
\]

2. Let \( y[n] \) be the amount of money in Mr. Rich’s bank account on the first day of the \( n^{th} \) month. Then, the factors which contribute to the change from \( y[n - 1] \) to \( y[n] \) are as follows:

- The interest earned based on the amount of money in the bank account on the fifteenth day of the \((n - 1)^{th}\) month: Since Mr. Rich does not withdraw or deposit any money after the first day and before the fifteenth day of any month, the amount of money in the bank account on the fifteenth day of any month is equal to the amount of money in the bank account on the first day of that month. Hence, the amount of interest earned is \( 0.01y[n - 1] \).

- The amount of money (\$500) that Mr. Rich withdraws on the last day of the \((n - 1)^{th}\) month.

Therefore, the difference equation relating \( y[n] \) to \( y[n - 1] \) is

\[
y[n] = 1.01y[n - 1] - x[n], \tag{9}
\]

where \( x[n] = 500 \) for each \( n \). We are given that the amount of money in the bank account on January 1, 2005 is \$10000. This represents the initial condition. Since our theoretical derivations have used \( n = 0 \) as the starting time with the initial conditions specified for \( n < 0 \), it is convenient to consider January 2005 as time \( n = -1 \) (note that the
system (9) is LTI so that we can choose any convenient time to be \( n = 0 \). We need to find the amount in the bank account on January 1, 2006, i.e., we need to find \( y[11] \). Hence, the difference equation formulation of the problem is:

Given the initial condition \( y[-1]=10000 \), solve the difference equation

\[
y[n] = 1.01y[n - 1] - x[n]
\]

with the input signal \( x[n] = 500 \) for all \( n \geq 0 \) and compute \( y[11] \). We can solve the difference equation (10) using either of the following two methods:

**Method 1 (Using one-sided \( z \)-transform):** Let the one-sided \( z \)-transforms of \( x[n] \) and \( y[n] \) be \( X(z) \) and \( Y(z) \), respectively, i.e.,

\[
X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \\
Y(z) = \sum_{n=0}^{\infty} y[n]z^{-n}.
\]

Recall that for one-sided \( z \)-transforms, we have the identity

\[
z\{y[n - n_0]\} = z^{-n_0}Y(z) + z^{-n_0}\sum_{n_1=-\infty}^{-1}y[n_1]z^{-n_1}.
\]

Hence,

\[
z\{y[n - 1]\} = z^{-1}Y(z) + z^{-1}y[-1]z = z^{-1}Y(z) + y[-1]. \tag{13}
\]

Taking the one-sided \( z \)-transform of both sides of (10) and using (13),

\[
Y(z) = 1.01(z^{-1}Y(z) + y[-1]) - X(z). \tag{14}
\]

Hence,

\[
(1 - 1.01z^{-1})Y(z) = 1.01y[-1] - X(z) \tag{15}
\]

\[
\Rightarrow Y(z) = \frac{1.01y[-1]}{(1 - 1.01z^{-1})} - \frac{X(z)}{(1 - 1.01z^{-1})} = \frac{(1.01)(10000)}{(1 - 1.01z^{-1})} - \frac{X(z)}{(1 - 1.01z^{-1})}. \tag{16}
\]
The first term on the right hand side of (16) is the homogeneous response (response due to initial conditions) while the second term on the right hand side of (16) is the forced response (response due to input signal). To obtain the signal $y[n]$, we need to take the inverse $z$-transform of (16). Since the input signal is $x[n] = 500$ for all $n \geq 0$, i.e., $x[n] = 500u[n]$, the $z$-transform (both one-sided and two-sided) of $x[n]$ is

$$X(z) = \frac{500}{1 - z^{-1}}. \quad (17)$$

To take the inverse $z$-transform of the second term in (16), we can use the partial fraction expansion

$$\frac{1}{(1 - z^{-1})(1 - 1.01z^{-1})} = \frac{101}{(1 - 1.01z^{-1})} - \frac{100}{(1 - z^{-1})}. \quad (18)$$

Noting that the original difference equation (10) is causal, both terms on the right hand side of (16) yield causal components when we take the inverse $z$-transform. Hence,

$$y[n] = (1.01)(10000)(1.01)^nu[n] - 500\left(101(1.01)^nu[n] - 100u[n]\right)$$

$$= -40400(1.01)^nu[n] + 50000u[n] \quad (19)$$

Evaluating at $n = 11$, we have

$$y[11] = 4927. \quad (20)$$

Hence, the amount of money in the bank account on January 1, 2006 is $4927.

A point to think about: What happens to $y[n]$ as $n \rightarrow \infty$? What does this physically mean in the context of our original problem which involves money in a bank account?

**Method 2 (By guessing exponentials):** From the difference equation (10), we know that the pole of the system is at 1.01. Since the input signal is $x[n] = 500u[n]$, the pole introduced by the input signal is at 1. Equivalently, we can see from (16) that the poles of $Y(z)$ are at 1 and 1.01. Furthermore, both these poles are simple poles, i.e.,

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they are not repeated poles. Hence, the solution $y[n]$ should be a linear combination of the two exponential signals $(1)^nu[n]$ and $(1.01)^nu[n]$. Let

$$y[n] = c_1(1)^nu[n] + c_2(1.01)^nu[n].$$

To solve for $c_1$ and $c_2$, we need to generate two equations. This can be done by numerically iterating the difference equation (10) for $n = 0$ and $n = 1$ to obtain

$$y[0] = 1.01y[-1] - 500 = 1.01 \times 10000 - 500 = 9600$$
$$y[1] = 1.01y[0] - 500 = 1.01 \times 9600 - 500 = 9196. \quad (22)$$

This yields the two equations:

$$(1)^0c_1 + (1.01)^0c_2 = 9600$$
$$(1)^1c_1 + (1.01)^1c_2 = 9196. \quad (23)$$

Solving the equations above for $c_1$ and $c_2$, we get $c_1 = 50000$ and $c_2 = -40400$. Hence, from (21)

$$y[n] = 50000(1)^nu[n] - 40400(1.01)^nu[n]. \quad (24)$$

As expected, the answer above matches the answer we obtained in Method 1.

The pole of the system (10) is at 1.01. Recall that the condition for BIBO stability is that all closed-loop poles should be inside the unit circle. Hence, the given system is not BIBO stable.

3. (a) The sampling period is picked to be $T_s = 0.01$ seconds. Since $y[n] \overset{\triangle}{=} y(nT_s)$, we find that $y[10 \times 100] = y(100 \times 100 \times 0.01) = y(10)$. Hence, finding $y(t)$ at $t = 10$ seconds is equivalent to finding $y[n]$ at $n = 10 \times 100$.

(b) In the differential equation

$$\frac{d}{dt}y = -y + x, \quad (25)$$

we are approximating $\frac{d}{dt}y$ using

$$\frac{dy(nT_s)}{dt} \approx \frac{y(nT_s) - y((n - 1)T_s)}{T_s}. \quad (26)$$
Hence,
\[
\frac{y[n] - y[n-1]}{T_s} = -y[n-1] + x[n-1].
\] (27)

This yields the difference equation
\[
y[n] = (1 - T_s)y[n-1] + T_s x[n-1]
\] (28)
\[
\implies y[n] = 0.99y[n-1] + 0.01x[n-1].
\] (29)

(c) The initial condition is given to be \(y(0) = 1\). Since \(y[n] \triangleq y(nT_s)\), the equivalent initial condition for the difference equation approximation is \(y[0] = 1\). The input signal is given to be \(x(t) = t\). Hence, the sampled input signal is \(x[n] \triangleq x(nT_s) = nT_s = 0.01n\).

(d) As in Problem 2, there are two methods to solve the difference equation (29), the first method using one-sided \(z\)-transforms, and the second method using the method of guessing exponentials. In this case, the method of guessing exponentials is significantly simpler. This is partly because, as we will see below, the input signal introduces a repeated pole so that the partial fraction expansion required by the method using one-sided \(z\)-transforms would be messy.

Since the input signal is \(x[n] = 0.01n\) for all \(n \geq 0\), i.e., \(x[n] = 0.01nu[n]\), the \(z\)-transform of the input signal is
\[
X(z) = 0.01 \frac{z^{-1}}{(1 - z^{-1})^2}.
\] (30)

Recall that the \(z\)-transform of \(nu[n]\) was worked out in Homework 4 to be \(\frac{z^{-1}}{(1 - z^{-1})^2}\).

From (30), the input signal introduces a repeated pole at 1. From (29), the system itself has a pole at 0.99. Hence, the output signal \(y[n]\) must be of the form
\[
y[n] = c_1(1)^nu[n] + c_2n(1)^n u[n] + c_3(0.99)^n u[n].
\] (31)

To find the constants \(c_1, c_2,\) and \(c_3\), we need to generate three equations. This can be done by numerically iterating the difference equation (29) for \(n = 1, n = 2,\) and \(n = 3\). Remember that
the value of $y[n]$ at $n = 0$ cannot be used as one of the equations to find $c_1$, $c_2$, and $c_3$ because $y[0]$ represents the initial condition (recall that when $y[-1]$ is the given initial condition, then we use $y[0]$, $y[1]$, etc. to solve for the coefficients $c_1$, $c_2$, etc.). We have

$$y[1] = 0.99y[0] + 0.01x[0] = 0.99 \times 1 + 0 = 0.99$$
$$y[2] = 0.99y[1] + 0.01x[1] = 0.99 \times 0.99 + 0.01 \times 0.01 = 0.9802$$
$$y[3] = 0.99y[2] + 0.01x[2] = 0.99 \times 0.9802 + 0.01 \times 0.02 = 0.970598.$$

Using (31), this yields the equations

$$c_1(1)^1 + c_2(1)(1)^1 + c_3(0.99)^1 = 0.99$$
$$c_1(1)^2 + c_2(2)(1)^2 + c_3(0.99)^2 = 0.9802$$
$$c_1(1)^3 + c_2(3)(1)^3 + c_3(0.99)^3 = 0.970598. \quad (32)$$

Solving the equations above, we get $c_1 = -1$, $c_2 = 0.01$, and $c_3 = 2$. Hence,

$$y[n] = -1(1)^n u[n] + 0.01n(1)^n u[n] + 2(0.99)^n u[n]$$
$$= (-1 + 0.01n + 2(0.99)^n)u[n]. \quad (33)$$

Evaluating at $n = 10 \times 100$, we obtain

$$y[10 \times 100] = 9.000086. \quad (34)$$

Note that we have approximately solved a differential equation by using difference equation techniques. It would be interesting to see how accurate our solution is. The original differential equation $\frac{d}{dt}y = -y + x$ can be solved in closed form given the input $x(t) = t$. Using, for instance, the method of integrating factors, we have

$$e^t \frac{d}{dt} y = -e^t y + e^t x \quad (35)$$

so that

$$\frac{d}{dt} (e^t y) = e^t x = e^t t. \quad (36)$$

Hence,

$$e^t y(t) - y(0) = \int_0^t e^\tau \tau d\tau. \quad (37)$$

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Using the method of integration by parts, we have

\[ \int_0^t e^\tau \tau d\tau = (te^t - e^t) - (0 - e^0) = te^t - e^t + 1. \quad (38) \]

Using (37),

\[ e^t y(t) = y(0) + te^t - e^t + 1 = 2 + te^t - e^t. \quad (39) \]

This yields

\[ y(t) = 2e^{-t} + t - 1. \quad (40) \]

Evaluating at \( t = 10 \), we have

\[ y(10) = 9.000091. \quad (41) \]

The difference between the exact solution obtained through the solution of the differential equation above and the approximate solution (34) can be thought of as the numerical error due to sampling the differential equation. It can be expected that if the sampling period \( T_s \) is made smaller, then the error reduces. In fact, it can be proved that given any error tolerance, the sampling period can be made small enough to make the error smaller than the tolerance value.