1. (a) The difference equation of the system is given to be
\[ y[n] = -y[n - 1] - 0.5y[n - 2] + x[n]. \] (1)
Taking the z-transform of both sides, we obtain
\[ Y(z) = -z^{-1}Y(z) - 0.5z^{-2}Y(z) + X(z). \] (2)
Hence, the transfer function is
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + z^{-1} + 0.5z^{-2}}. \] (3)
Substituting \( z = e^{j\omega} \), the frequency response of the system is
\[ H(e^{j\omega}) = \frac{1}{1 + e^{-j\omega} + 0.5e^{-2j\omega}}. \] (4)
(b) The input signal is given to be \( x[n] = \sin(\frac{n\pi}{3} - \frac{1}{2})u[n] \) which can be rewritten as
\[ x[n] = \frac{1}{2j} \left[ e^{j\left(\frac{\pi}{3}n - \frac{1}{2}\right)} - e^{-j\left(\frac{\pi}{3}n - \frac{1}{2}\right)} \right] u[n]. \] (5)
Evaluating (4) at \( \omega = \frac{\pi}{3} \) and \( \omega = -\frac{\pi}{3} \) gives
\[ H(e^{j\frac{\pi}{3}}) = 0.3846 + 0.3997j = 0.5547e^{0.8046j}, \] (6)
\[ H(e^{-j\frac{\pi}{3}}) = 0.3846 - 0.3997j = 0.5547e^{-0.8046j}. \] (7)
Hence, from (5), the output signal at steady-state is
\[ y[n] = \frac{1}{2j} \left[ |H(e^{j\frac{\pi}{3}})|e^{j\angle H(e^{j\frac{\pi}{3}})}e^{j\left(\frac{\pi}{3}n - \frac{1}{2}\right)} - |H(e^{-j\frac{\pi}{3}})|e^{j\angle H(e^{-j\frac{\pi}{3}})}e^{-j\left(\frac{\pi}{3}n - \frac{1}{2}\right)} \right] u[n] \]
\[ = \frac{1}{2j} \left[ 0.5547e^{j\angle H(e^{j\frac{\pi}{3}})}e^{j\left(\frac{\pi}{3}n - \frac{1}{2}\right)} - 0.5547e^{-j\angle H(e^{-j\frac{\pi}{3}})}e^{-j\left(\frac{\pi}{3}n - \frac{1}{2}\right)} \right] u[n] \]
which can be simplified to
\[ y[n] = 0.5547 \sin(\frac{\pi}{3}n - \frac{1}{2} + 0.8046)u[n] \]
\[ = 0.5547 \sin(\frac{\pi}{3}n + 0.3046)u[n]. \] (8)
Equivalently, noting that the frequency of the input signal is $\frac{\pi}{3}$ and that, from (4), we expect a magnitude gain of $|H(e^{j\frac{\pi}{3}})| = 0.5547$ and a phase shift of $\angle H(e^{j\frac{\pi}{3}}) = 0.8046$, the output signal at steady-state is

$$y[n] = |H(e^{j\frac{\pi}{3}})| \sin\left(\frac{\pi}{3} n - \frac{1}{2} + \angle H(e^{j\frac{\pi}{3}})\right) u[n]$$

$$= 0.5547 \sin\left(\frac{\pi}{3} n + 0.3046\right) u[n]. \quad (9)$$

2. (a) $X(e^{j\omega}) = e^{j\omega} + e^{2j\omega}$: Substituting $e^{j\omega} = z$, we get $X(z) = z + z^2$. Hence, taking the causal inverse $z$-transform of $X(z)$, we get

$$x[n] = \delta[n + 1] + \delta[n + 2]. \quad (10)$$

(b) $X(e^{j\omega}) = \frac{e^{j\omega}}{1-e^{-j\omega}}$: Substituting $e^{j\omega} = z$, we get

$$X(z) = \frac{z}{1 - z^{-2}} \quad (11)$$

Let $X_1(z) = \frac{1}{1 - z^{-2}}$. Hence, $X(z) = zX_1(z)$. If we find the inverse $z$-transform of $X_1(z)$ to be $x_1[n]$, then the inverse $z$-transform of $X(z)$ is given by $x[n] = x_1[n + 1]$ because $X(z) = zX_1(z)$. $X_1(z)$ can be decomposed into partial fractions as

$$X_1(z) = \frac{0.5}{1 - z^{-1}} + \frac{0.5}{1 + z^{-1}}. \quad (12)$$

Taking the causal inverse $z$-transform of $X_1(z)$, we get

$$x_1[n] = 0.5(1)^n u[n] + 0.5(-1)^n u[n]$$

$$= 0.5u[n] + 0.5(-1)^n u[n]. \quad (13)$$

Hence,

$$x[n] = x_1[n + 1] = 0.5u[n + 1] + 0.5(-1)^{n+1} u[n + 1]. \quad (14)$$

Alternatively, if we take the partial fraction expansion of $X(z)$ directly, we get

$$X(z) = \frac{1}{z^{-1}} + \frac{0.5}{1 - z^{-1}} - \frac{0.5}{1 + z^{-1}}. \quad (15)$$
so that
\[ x[n] = \delta[n + 1] + 0.5u[n] - 0.5(-1)^nu[n]. \]  \hspace{1cm} (16)
Using \( u[n + 1] = u[n] + \delta[n + 1] \), it can be easily shown that the
two answers we got above in (14) and (16) are identical.

(c) \( X(e^{j\omega}) = \sin(\omega) \): We know that \( \sin(\omega) = \frac{1}{2j}(e^{j\omega} - e^{-j\omega}) \). Hence,
substituting \( e^{j\omega} = z \), we get
\[ X(z) = \frac{1}{2j}(z - z^{-1}). \]  \hspace{1cm} (17)
Hence, taking the causal inverse \( z \)-transform of \( X(z) \), we find
\[ x[n] = \frac{1}{2j}(\delta[n + 1] - \delta[n - 1]). \]  \hspace{1cm} (18)

3. We know that if the DTFT of \( x[n] \) is \( X(e^{j\omega}) \), then the DTFT of \( x^*[\cdot] \) is \( X^*(e^{j\omega}) \). Refer to the lecture notes on Discrete-Time Fourier Transform for the proof of this fact.

(a) The even part of \( x[n] \) is defined as
\[ x_e[n] = \frac{x[n] + x^*[\cdot]n}{2}. \]  \hspace{1cm} (19)
Hence, the DTFT of \( x_e[n] \) is
\[ X_e(e^{j\omega}) = \frac{X(e^{j\omega}) + X^*(e^{j\omega})}{2} = \text{Re}\{X(e^{j\omega})\}. \]  \hspace{1cm} (20)

(b) The odd part of \( x[n] \) is defined as
\[ x_o[n] = \frac{x[n] - x^*[\cdot]n}{2}. \]  \hspace{1cm} (21)
Hence, the DTFT of \( x_o[n] \) is
\[ X_o(e^{j\omega}) = \frac{X(e^{j\omega}) - X^*(e^{j\omega})}{2} = j\text{Im}\{X(e^{j\omega})\}. \]  \hspace{1cm} (22)