EE3054 - Homework 8 - Solutions

1. (a) \( y(t) = x(t) + x(t + 1) \): This system is linear since if \( x_1(t) \mapsto y_1(t) \) and \( x_2(t) \mapsto y_2(t) \), then \((\alpha x_1(t) + \beta x_2(t)) \mapsto (\alpha y_1(t) + \beta y_2(t))\) for any \( \alpha \) and \( \beta \). The system is noncausal since the value of the output signal at time \( t \) depends on the value of the input signal at time \( t + 1 \). The system is time invariant since if \( x(t) \mapsto y(t) \), then \( x(t-t_0) \mapsto y(t-t_0) \) for any \( t_0 \). This can also be inferred simply by noting that \( t \) does not appear separately in the system equation.

(b) \( \frac{dy}{dt} = x^3 \): This system is nonlinear due to the \( x^3 \) term. The system is causal since the value of the output signal at any time does not depend on the values of the input signal at future times. The system is time invariant since if \( x(t) \mapsto y(t) \), then \( x(t-t_0) \mapsto y(t-t_0) \) for any \( t_0 \).

(c) \( \frac{d^2y}{dt^2} + \frac{dy}{dt} = x \): This system is linear since if \( x_1(t) \mapsto y_1(t) \) and \( x_2(t) \mapsto y_2(t) \), then \( (\alpha x_1(t) + \beta x_2(t)) \mapsto (\alpha y_1(t) + \beta y_2(t)) \) for any \( \alpha \) and \( \beta \). The system is causal since the value of the output signal at any time does not depend on the values of the input signal at future times. The system is time invariant since if \( x(t) \mapsto y(t) \), then \( x(t-t_0) \mapsto y(t-t_0) \) for any \( t_0 \).

2. The step response of a continuous-time system is given to be \( s(t) = e^{-t}u(t) \). We know that the derivative of the unit step is the unit impulse, i.e., \( \delta(t) = \frac{du(t)}{dt} \). Hence, the impulse response \( h(t) \) is the derivative of the step response, i.e.,

\[
\begin{align*}
    h(t) & = \frac{ds(t)}{dt} \\
          & = \frac{d}{dt}(e^{-t}u(t)) \\
          & = -e^{-t}u(t) + e^{-t}\delta(t) \quad (1)
\end{align*}
\]

where in the last step, we have used the fact that \( \delta(t) = \frac{du(t)}{dt} \). Using the sampling property of the unit impulse which says that \( x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \), we have \( e^{-t}\delta(t) = e^{-0}\delta(t) = \delta(t) \). Hence, the impulse response obtained in (1) simplifies to

\[
    h(t) = -e^{-t}u(t) + \delta(t). \quad (2)
\]
To check BIBO stability of the system, we need to verify if \( \int_{-\infty}^{\infty} |h(t)| dt < \infty \). Using the impulse response we calculated above, we have

\[
\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} | - e^{-t}u(t) + \delta(t) | dt \\
\leq \int_{-\infty}^{\infty} | - e^{-t}u(t) | dt + \int_{-\infty}^{\infty} \delta(t) dt \\
= \int_{0}^{\infty} e^{-t} dt + 1 \\
= 1 + 1 = 2 < \infty. \tag{3}
\]

Hence, the system is BIBO stable.

3. (a) \( x_1(t) = e^{-(t-4)}u(t-4)\delta(t-5) \):

From the sampling property of the unit impulse, we know that \( x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \). Hence, the given signal simplifies to \( x_1(t) = e^{-(5-4)}u(5-4)\delta(t-5) = \frac{1}{e}\delta(t-5) \).

(b) \( x_2(t) = \int_{-\infty}^{t-5} \delta(\tau - 1) d\tau \):

Using the dummy variable \( \tau_1 = \tau - 1 \), the given signal is equal to \( \int_{-\infty}^{t-6} \delta(\tau_1) d\tau_1 \). We know that the unit step is the integral of the unit impulse, i.e., \( u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \). Hence, the given signal is the same as \( u(t-6) \).

(c) \( x_3(t) = \frac{d}{dt} \{ e^{-(t-4)}u(t-4) \} \):

Expanding the derivative, we get

\[
x_3(t) = -e^{-(t-4)}u(t-4) + e^{-(t-4)}\delta(t-4). \tag{4}
\]

Using the sampling property of the unit impulse which says that \( x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \), the given signal \( x_3(t) \) simplifies to

\[
x_3(t) = -e^{-(t-4)}u(t-4) + \delta(t-4). \tag{5}
\]

(d) \( x_4(t) = \delta(t-1) * \delta(t-2) * \delta(t) \):

We know that the convolution of any signal \( x(t) \) with the unit impulse is the same signal \( x(t) \), i.e., \( x(t) * \delta(t) = x(t) \). Hence, we have

\[
\delta(t-1) * \delta(t) = \delta(t-1). \tag{6}
\]
More generally, we know that $x(t) * \delta(t - t_0) = x(t - t_0)$ for any signal $x(t)$ and any $t_0$. Hence,

$$\delta(t - 2) * \delta(t - 1) = \delta(t - 3). \quad (7)$$

The given signal is $x_4(t) = \delta(t-1) * \delta(t-2) * \delta(t)$. Since convolution is commutative and associative, we can evaluate $x_4(t)$ as follows:

$$x_4(t) = \delta(t - 2) * (\delta(t - 1) * \delta(t)). \quad (8)$$

Using (6) and (7), (8) simplifies to

$$x_4(t) = \delta(t - 2) * \delta(t - 1)$$
$$= \delta(t - 3). \quad (9)$$