1. The impulse response of the system is given to be \( h(t) = e^{-2t}u(t) \). Given an input signal \( x(t) \), the output signal is \( y(t) = h(t) \ast x(t) = x(t) \ast h(t) \).

(a) \( x(t) = u(t) \): The output signal is
\[
y(t) = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau) u(t - \tau) \, d\tau.
\]
Note that \( u(\tau) = 0 \) for \( \tau < 0 \) and \( u(t - \tau) = 0 \) for \( \tau > t \). Hence, if \( t < 0 \), the product of \( u(\tau) \) and \( u(t - \tau) \) is zero for all \( \tau \) implying that \( y(t) = 0 \) for \( t < 0 \). If \( t > 0 \), noting that \( u(\tau)u(t - \tau) \) is nonzero only for \( \tau \) in the interval \([0, t]\), the limits of integration in (1) can be reduced to
\[
y(t) = \int_{0}^{t} e^{-2\tau} \, d\tau = -\frac{1}{2} e^{-2t} \bigg|_{0}^{t} = \frac{1 - e^{-2t}}{2}.
\]
Hence, the output signal \( y(t) \) is zero for \( t < 0 \) and \( \frac{1 - e^{-2t}}{2} \) for \( t \geq 0 \). Therefore, the output signal is
\[
y(t) = \frac{1 - e^{-2t}}{2} u(t).
\]

(b) \( e^t u(t) \): Using the same procedure as in part (a), we can conclude that \( y(t) \) is zero for \( t < 0 \). For \( t \geq 0 \),
\[
y(t) = \int_{0}^{t} e^{-2\tau} e^{t - \tau} \, d\tau
\]
\[
= e^t \left( -\frac{1}{3} e^{-3\tau} \bigg|_{0}^{t} \right)
\]
\[
= e^t \frac{1 - e^{-3t}}{3} = \frac{e^t - e^{-2t}}{3}.
\]
Hence, the output signal \( y(t) \) is zero for \( t < 0 \) and \( \frac{e^t - e^{-2t}}{3} \) for \( t \geq 0 \). Therefore, the output signal is
\[
y(t) = \frac{e^t - e^{-2t}}{3} u(t).
\]
(c) $e^{-t}u(t)$: Using the same procedure as in part (a), we can conclude that $y(t)$ is zero for $t < 0$. For $t \geq 0$,

$$y(t) = \int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau$$

$$= e^{-t}\left(-e^{-t}\bigg|_0^t\right)$$

$$= e^{-t}(1-e^{-t})$$

$$= e^{-t} - e^{-2t}. \quad (6)$$

Hence, the output signal $y(t)$ is zero for $t < 0$ and $e^{-t} - e^{-2t}$ for $t \geq 0$. Therefore, the output signal is

$$y(t) = (e^{-t} - e^{-2t})u(t). \quad (7)$$

2. The impulse response of the system is given to be $h(t) = \delta(t) + 2^{-t}u(t)$.

(a) Since $h(t) = 0$ for all $t < 0$, the system is causal.

(b) To decide whether the system is BIBO stable, we have to check if $\int_{-\infty}^{\infty} |h(t)|dt$ is finite. From the given impulse response, we find that

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} [\delta(t) + 2^{-t}u(t)]dt$$

$$= 1 + \int_0^{\infty} 2^{-t}dt$$

$$= 1 - \frac{2^{-t}}{\ln(2)} \bigg|_0^{\infty}$$

$$= 1 + \frac{1}{\ln(2)}. \quad (8)$$

Hence, $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ implying that the system is BIBO stable.

(c) We know that the step response $s(t)$ is the integral of the impulse response, i.e., $s(t) = \int_{-\infty}^{t} h(\tau)d\tau$. For any $t < 0$, $\int_{-\infty}^{t} h(\tau)d\tau$ is zero since $h(t) = 0$ for all $t < 0$. If $t \geq 0$, $s(t)$ is given by

$$s(t) = \int_{-\infty}^{t} [\delta(\tau) + 2^{-\tau}u(\tau)]d\tau$$

2
\[
= \left. u(t) + \int_0^t 2^{-\tau}d\tau \right|

= u(t) - \frac{2^{-t} - 1}{\ln(2)}u(t).
\]

(9)

Hence, \(s(t)\) is zero for \(t < 0\) and \(u(t) - \frac{2^{-t} - 1}{\ln(2)}u(t)\) for \(t \geq 0\).

Therefore, the step response of the system is

\[
s(t) = \left[1 - \frac{2^{-t} - 1}{\ln(2)}\right]u(t).
\]

(10)

3. The relation between the input signal \(x(t)\) and the output signal \(y(t)\) of a continuous-time LTI system is given to be

\[
y(t) = \int_0^0 x(t - \tau)d\tau.
\]

(11)

Using the dummy variable \(t_1 = t - \tau\), we can also write the relation between \(x(t)\) and \(y(t)\) as

\[
y(t) = -\int_{t+1}^t x(t_1)dt_1 = \int_t^{t+1} x(t_1)dt_1
\]

(12)

(a) From (12), we see that \(y(t)\) depends on values of the input signal on the time interval \([t, t+1]\). Hence, the value of the output signal at any time depends on future values of the input signal. Therefore, the system is not causal.

(b) We can use two methods to check BIBO stability of the system.

**Method 1:** From the definition of BIBO stability, we want to check if we can demonstrate that the output signal is bounded if we are given that the input signal is bounded. Let \(M_x\) be a magnitude bound on the input signal, i.e., \(|x(t)| \leq M_x\) for all \(t\). From (12), this implies that

\[
|y(t)| \leq \int_t^{t+1} M_x dt_1 = M_x.
\]

(13)
Hence, the output signal is also bounded with the same magnitude bound $M_x$. Therefore, the output signal is bounded whenever the input signal is bounded. This means that the system is BIBO stable.

**Method 2:** We can also check BIBO stability by finding the impulse response and computing $\int_{-\infty}^{\infty} |h(t)|dt$. From (12), the impulse response can be found by substituting $x(t) = \delta(t)$ to obtain

$$y(t) = \int_{t}^{t+1} \delta(t_1)dt_1. \quad (14)$$

Since $\delta(t_1)$ is nonzero only at $t_1 = 0$, we can see from (14) that if $t > 0$, then $y(t) = 0$ because the interval $[t, t+1]$ does not include 0 when $t > 0$. Similarly when $t < -1$, we have $y(t) = 0$. But, if $t$ is bigger than -1 but smaller than 0, then the interval $(t, t+1)$ includes 0 so that $y(t) = 1$. If $t = -1$ or $t = 0$, then $y(t) = \frac{1}{2}$ since the integral in (14) would only include half of the unit impulse. Therefore,

$$h(t) = \begin{cases} 1 & \text{if } t \in (-1, 0) \\ \frac{1}{2} & \text{if } t = -1 \text{ or } t = 0 \\ 0 & \text{if } t < -1 \text{ or } t > 0. \end{cases} \quad (15)$$

Since $h(t)$ is nonzero for some values of negative $t$, this system is not causal. Also, $\int_{-\infty}^{\infty} |h(t)|dt = 1 < \infty$. Hence, the system is BIBO stable.

(c) The impulse response of the system was found above to be as in (15).

(d) The step response can be computed by integrating the impulse response. Hence, the step response $s(t)$ is given by

$$s(t) = \int_{-\infty}^{t} h(t)dt$$

$$= \begin{cases} 0 & \text{if } t \leq -1 \\ 1 + t & \text{if } t \in (-1, 0) \\ 1 & \text{if } t \geq 0. \end{cases} \quad (16)$$