1. (6 points) You are given a system with the impulse response \( h(t) = 2^{-2t}u(t) \).

(a) What is the output of the system when the input is \( x(t) = \delta(t + 1) \)?
(b) Find the step response of the system.
(c) Is this system BIBO stable?

Solution:

(a) Given the input \( x(t) = \delta(t + 1) \), the output \( y(t) \) is \( y(t) = x(t) * h(t) = \delta(t + 1) * h(t) \). We know that \( x(t) * \delta(t - t_0) = x(t - t_0) \) for any signal \( x(t) \) and any time shift \( t_0 \). Therefore, for the given input signal, the output signal is \( y(t) = h(t + 1) = 2^{-2(t+1)}u(t+1) \).

(b) Given the impulse response of the system, the step response is

\[
s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \int_{-\infty}^{t} 2^{-2\tau}u(\tau)d\tau = \int_{0}^{t} 2^{-2\tau}d\tau u(t)
= \frac{1 - 2^{-2t}}{2 \ln(2)} u(t). \tag{1}
\]

(c) To check BIBO stability, we have to verify if \( \int_{-\infty}^{\infty} |h(\tau)|d\tau \) is finite. For the given \( h(t) \), we have

\[
\int_{-\infty}^{\infty} |h(\tau)|d\tau = \int_{-\infty}^{\infty} 2^{-2\tau}u(\tau)d\tau = \int_{0}^{\infty} 2^{-2\tau}d\tau = \frac{1}{2 \ln(2)} < \infty. \tag{2}
\]

Hence, the system is BIBO stable.
2. (4 points) For each of the following systems, indicate if the system is linear/nonlinear, causal/noncausal, and time varying/time invariant. Briefly, explain your reasoning.

(a) \( \ddot{y}(t) + 2ty(t) = x(t) \)

(b) (For extra credit: 4 points) \( y(t) = \int_0^2 x(t-\tau^2)d\tau \)

Solution:

(a) The given system is **linear** since \( x \) and \( y \) both appear linearly in the given differential equation. More rigorously, we can show that if \( x(t) \mapsto y(t) \), then \( \alpha x(t) \mapsto \alpha y(t) \) for any real number \( \alpha \). Also, if \( x_1(t) \mapsto y_1(t) \) and \( x_2(t) \mapsto y_2(t) \), then \( (x_1(t) + x_2(t)) \mapsto (y_1(t) + y_2(t)) \).

The given system is **causal** since the value of the output signal at any time \( t_0 \) does not depend on values of the input signal for times larger than \( t_0 \), i.e., the value of the output signal at any time does not depend on future values of the input signal. The given system is **time varying** because of the coefficient \( t \) in the \( 2ty(t) \) term. More rigorously, let \( x(t) \mapsto y(t) \) and let \( t_0 \) be a constant. Then, by the definition of the system, we have \( \ddot{y}(t) + 2ty(t) = x(t) \) for all values of time \( t \). In particular, therefore, \( \ddot{y}(t-t_0) + 2(t-t_0)y(t-t_0) = x(t-t_0) \). Therefore, it is not true that \( x(t-t_0) \mapsto y(t-t_0) \) since

\[
\ddot{y}(t-t_0) + 2(t-t_0)y(t-t_0) \neq \ddot{y}(t-t_0) + 2ty(t-t_0).
\]

(b) The given system is **linear** since \( x \) and \( y \) both appear linearly in the given differential equation. More rigorously, we can show that if \( x(t) \mapsto y(t) \), then \( \alpha x(t) \mapsto \alpha y(t) \) for any real number \( \alpha \). Also, if \( x_1(t) \mapsto y_1(t) \) and \( x_2(t) \mapsto y_2(t) \), then \( (x_1(t) + x_2(t)) \mapsto (y_1(t) + y_2(t)) \).

**Note:** Do not get confused by the \( \tau^2 \) term. \( \tau \) is a dummy variable in the integral. Linearity depends on the input-output relation, i.e., the relation between \( x \) and \( y \).

The given system is **causal** since the value of the output signal at any time \( t_0 \) does not depend on values of the input signal for times larger than \( t_0 \), i.e., the value of the output signal at any time does not depend on future values of the input signal. From the given integral relation \( y(t) = \int_0^2 x(t-\tau^2)d\tau \), it is clear that \( y(t_0) \) depends on values of \( x(t) \) in the interval \([t_0 - 4, t_0]\).

The given system is **time invariant** because if \( x(t) \mapsto y(t) \), then \( x(t-t_0) \mapsto y(t-t_0) \) for any \( t_0 \). This can also be seen by simply noting that the given integral relation \( y(t) = \int_0^2 x(t-\tau^2)d\tau \) does not involve \( t \) explicitly.