1. (8 points) Consider the system

\[ y[n] = y[n-1] + y[n-2] + x[n]. \]

(a) Is this system LTI? Is this system FIR or IIR? Briefly justify your answers.

(b) Find the transfer function \( H(z) \) of the system. Find the poles of the system.

**Solution:**

(a) We know that an LTI IIR system is of the general form

\[ y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]. \]

Hence, the given system is of the form of an LTI IIR system. Note that the right hand side of the given difference equation has both \( x \) and \( y \) terms. Therefore, the given system is LTI and IIR.

(b) Taking the \( z \)-transform of both sides of the difference equation, we get

\[
Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z). \quad (1)
\]

Hence, the transfer function is

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} - z^{-2}}. \quad (2)
\]

The poles of the system are the roots of the quadratic equation

\[ 1 - z^{-1} - z^{-2} = 0. \]

Hence, the poles of the system are \(-0.618\) and \(1.618\).
2. (10 points) Consider the cascade interconnection of Systems 1 and 2 shown below.

The impulse response of System 1 is given to be \( h_1[n] = 2^n u[n] \) and the difference equation of System 2 is given to be \( y[n] = y_1[n] + y_1[n - 1] \) where, as shown in the figure, \( y_1[n] \) is the input signal for System 2.

(a) Find the transfer function of the overall system.
(b) Find the impulse response of the overall system.

Solution:

(a) Since the impulse response of System 1 is given to be \( h_1[n] = 2^n u[n] \), the transfer function of System 1 is

\[
H_1(z) = \sum_{n=-\infty}^{\infty} h_1[n] z^{-n} = \sum_{n=0}^{\infty} 2^n z^{-n} = \frac{1}{1 - 2z^{-1}}.
\]  

(3)

\( H_1(z) \) can also be immediately obtained from the general result that the \( z \)-transform of the signal \( \alpha^n u[n] \) is \( \frac{1}{1 - \alpha z^{-1}} \).

Since the difference equation of System 2 is given to be \( y[n] = y_1[n] + y_1[n - 1] \), the transfer function of System 2 is \( H_2(z) = 1 + z^{-1} \).

Hence, the transfer function of the overall system (which is the cascade interconnection of System 1 and System 2) is

\[
H(z) = H_1(z)H_2(z) = \frac{1 + z^{-1}}{1 - 2z^{-1}}.
\]  

(4)

(b) The impulse response \( h[n] \) of the overall system is the inverse \( z \)-transform of \( H(z) \). Rewriting \( H(z) \) as

\[
H(z) = -0.5 + \frac{1.5}{1 - 2z^{-1}}.
\]  

(5)
the impulse response $h[n]$ is obtained to be

$$h[n] = -0.5\delta[n] + 1.5(2)^n u[n]. \quad (6)$$

Note that both System 1 and System 2 are causal. Hence, the overall system is causal. Therefore, we have taken the causal inverse $z$-transform to find $h[n]$ above.

The impulse response can also be obtained more easily by just using the definition that the impulse response is the response of the system to a unit impulse. Hence, if the input signal $x[n]$ is taken to be the unit impulse $u[n]$, then the output of System 1 is $h_1[n] = 2^nu[n]$. Hence, the input to System 2 is $2^n u[n]$ so that (from the given difference equation of System 2), the output of System 2 is

$$h[n] = 2^n u[n] + 2^{n-1} u[n-1]. \quad (7)$$

Note: Using $u[n-1] = u[n] - \delta[n]$, you can show that the two answers we got above for $h[n]$ in (6) and (7) are identical.

3. (12 points) Given a function $H(e^{j\omega}) = \tan(\omega)$, we want to compute its inverse DTFT $h[n]$. Find $h[n]$ using the following steps:

(a) Show that $\tan(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{j(e^{j\omega} + e^{-j\omega})}$. Hence, show that $H(z) = \frac{z - z^{-1}}{j(z + z^{-1})}$.

(b) Show that $H(z)$ can be rewritten as

$$H(z) = \frac{1}{j} \left[ -1 + \frac{1}{1 + jz^{-1}} + \frac{1}{1 - jz^{-1}} \right].$$

(c) Show that

$$h[n] = \frac{1}{j} (\delta[n] + (-j)^n u[n] + (j)^n u[n]).$$

Solution:

(a) We know that $\cos(\omega) = \frac{1}{2} (e^{j\omega} + e^{-j\omega})$ and $\sin(\omega) = \frac{1}{2j} (e^{j\omega} - e^{-j\omega})$. Hence,

$$\tan(\omega) = \frac{\sin(\omega)}{\cos(\omega)} = \frac{e^{j\omega} - e^{-j\omega}}{j(e^{j\omega} + e^{-j\omega})}. \quad (8)$$

Substituting $z$ for $e^{j\omega}$, we get $H(z) = \frac{z - z^{-1}}{j(z + z^{-1})}$. 3
(b) Multiplying the numerator and denominator of $H(z)$ by $z^{-1}$, we get $H(z) = \frac{1 - z^{-2}}{1 + z^{-2}}$. Dividing the numerator by the denominator, we get

$$H(z) = \frac{1}{j} \left[ -1 + \frac{2}{1 + z^{-2}} \right]. \quad (9)$$

The poles of $H(z)$, i.e., the roots of the equation $1 + z^{-2} = 0$, are $+j$ and $-j$. Hence, the partial fraction expansion of the term $\frac{2}{1 + z^{-2}}$ will be of the form

$$\frac{2}{1 + z^{-2}} = \frac{A_1}{1 + jz^{-1}} + \frac{A_2}{1 - jz^{-1}}. \quad (10)$$

To find $A_1$ and $A_2$, multiply both sides of (10) by $(1 + z^{-2})$ and equate the constant terms and the coefficients of $z^{-1}$ on both sides of the resulting equation. This gives the two equations

$$A_1 + A_2 = 2$$
$$-jA_1 + jA_2 = 0. \quad (11)$$

Solving, we get $A_1 = 1$ and $A_2 = 1$. Hence, from (9),

$$H(z) = \frac{1}{j} \left[ -1 + \frac{1}{1 + jz^{-1}} + \frac{1}{1 - jz^{-1}} \right]. \quad (12)$$

(c) To find $h[n]$, take the (causal) inverse $z$-transform of $H(z)$. Hence,

$$h[n] = \frac{1}{j} \left( (-\delta[n] + (-j)^n u[n] + (j)^n u[n]) \right). \quad (13)$$

4. (12 points) You are given a system with difference equation

$$y[n] = 0.5y[n-1] + x[n].$$

(a) Find the frequency response $H(e^{j\omega})$ of the system. Find the magnitude response $|H(e^{j\omega})|$ and the phase response $\angle H(e^{j\omega})$.

(b) If the input signal is $x[n] = 2 \sin(\frac{\pi}{3} n + \frac{\pi}{3}) u[n]$, find the steady-state output signal.
Solution:

(a) Taking the $z$-transform of both sides of the given difference equation, we get

$$Y(z) = 0.5z^{-1}Y(z) + X(z). \quad (14)$$

Hence, the transfer function of the system is

$$H(z) = \frac{1}{1 - 0.5z^{-1}}. \quad (15)$$

Substituting $z = e^{j\omega}$, the frequency response of the system is

$$H(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}. \quad (16)$$

From $H(e^{j\omega})$, the magnitude response and the phase response can be calculated as

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1 - 0.5\cos(\omega))^2 + 0.25\sin^2(\omega)}} \quad (17)$$

$$\angle H(e^{j\omega}) = -\tan^{-1}\left(\frac{0.5\sin(\omega)}{1 - 0.5\cos(\omega)}\right). \quad (18)$$

(b) The input signal is given to be $x[n] = 2\sin(\frac{\pi}{3}n + \frac{\pi}{3})u[n]$. The input signal can be rewritten as

$$x[n] = \frac{2}{2j} \left[ e^{j\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)} - e^{-j\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)} \right] u[n]. \quad (19)$$

Evaluating (17) and (18) at $\omega = \frac{\pi}{3}$ and $\omega = -\frac{\pi}{3}$ gives

$$|H(e^{j\frac{\pi}{3}})| = 1.1547 \quad \angle H(e^{j\frac{\pi}{3}}) = -\frac{\pi}{6}$$

$$|H(e^{-j\frac{\pi}{3}})| = 1.1547 \quad \angle H(e^{-j\frac{\pi}{3}}) = \frac{\pi}{6}. \quad (20)$$

Hence, from (19), the output signal at steady-state is

$$y[n] = \frac{2}{2j} \left[ |H(e^{j\frac{\pi}{3}})| e^{j\angle H(e^{j\frac{\pi}{3}})} e^{j\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)} - |H(e^{-j\frac{\pi}{3}})| e^{j\angle H(e^{-j\frac{\pi}{3}})} e^{-j\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)} \right] u[n]$$

$$= \frac{2}{2j} \left[ 1.1547 e^{-j\frac{\pi}{3}} e^{j\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)} - 1.1547 e^{j\frac{\pi}{3}} e^{-j\left(\frac{\pi}{3}n + \frac{\pi}{3}\right)} \right] u[n]$$
which can be simplified to

\[
y[n] = (2)(1.1547) \sin\left(\frac{\pi}{3}n + \frac{\pi}{3} - \frac{\pi}{6}\right) = 2.3094 \sin\left(\frac{\pi}{3}n + \frac{\pi}{6}\right). 
\]

Equivalently, noting that the frequency of the input signal is \(\frac{\pi}{3}\) and that, from (17) and (18), we expect a magnitude gain of \(|H(e^{j\frac{\pi}{3}})| = 1.1547\) and a phase shift of \(\angle H(e^{j\frac{\pi}{3}}) = -\frac{\pi}{6}\), the output signal is steady-state is

\[
y[n] = 2|H(e^{j\frac{\pi}{3}})| \sin\left(\frac{\pi}{3}n + \frac{\pi}{3} + \angle H(e^{j\frac{\pi}{3}})u[n]\right) 
\]

\[
= (2)(1.1547) \sin\left(\frac{\pi}{3}n + \frac{\pi}{3} - \frac{\pi}{6}\right) = 2.3094 \sin\left(\frac{\pi}{3}n + \frac{\pi}{6}\right).
\]

5. (8 points) The step response of an unknown system is observed to be \((\delta[n] + \delta[n - 1] + \delta[n - 2])\). Find the impulse response of the system (\textit{Hint:} the easy way to do this is by plotting \(u[n]\) and \(u[n - 1]\)). Also, find the equation relating the input signal \(x[n]\) and the output signal \(y[n]\) of the system.

**Solution:** In this problem, we are given that the step response is

\[
y_u[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]. \quad (21)
\]

We know that (you can easily verify this by plotting \(u[n]\) and \(u[n - 1]\))

\[
\delta[n] = u[n] - u[n - 1]. \quad (22)
\]

Hence, the impulse response is

\[
h[n] = y_u[n] - y_u[n - 1] 
\]

\[
= (\delta[n] + \delta[n - 1] + \delta[n - 2]) - (\delta[n - 1] + \delta[n - 2] + \delta[n - 3]) 
\]

\[
= \delta[n] - \delta[n - 3]. \quad (23)
\]

Given an arbitrary input signal \(x[n]\), the corresponding output signal is obtained as \(y[n] = x[n] * h[n] = x[n] * (\delta[n] - \delta[n - 3]) = x[n] - x[n - 3]\). Hence, the equation relating the input signal \(x[n]\) and the output signal \(y[n]\) of the system is

\[
y[n] = x[n] - x[n - 3]. \quad (24)
\]