A Time-Invariant Dual High-Gain Based Adaptive Output-Feedback Controller for Nonlinear Systems

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***Abstract*— We propose an adaptive output-feedback controller for a general class of nonlinear triangular (strict-feedback-like) systems. The design is based on our recent results on a dual high-gain observer and controller architecture with a dynamic scaling. The technique provides strong robustness properties and allows the system class to contain unknown functions dependent on all states and involving unknown parameters (with no magnitude bounds required). Unlike our earlier result on this problem where a time-varying design of the high-gain scaling parameter was utilized, the technique proposed here achieves an autonomous dynamic controller by introducing a novel design of the observer, the scaling parameter, and the adaptation parameter. This provides a time-invariant dynamic output-feedback controller for the benchmark open problem proposed in our earlier work with no magnitude bounds or sign information on the unknown parameter being necessary.**

I. INTRODUCTION

We consider the class of systems:

$$\begin{aligned} \dot{x}_i &= \phi_i(x_1,\ldots,x_i)+\phi_{(i,i+1)}(x_1)x_{i+1}\,,\ i=1,\ldots,n-1 \\ \dot{x}_n &= \phi_n(x_1,\ldots,x_n)+\mu_0(x_1)u \\ y &= x_1 \end{aligned} \tag{1}$$

where[1] $x=[x_1,\ldots,x_n]^T \in \mathcal{R}^n$ is the state, $y \in \mathcal{R}$ the output, and $u \in \mathcal{R}$ the input. $\phi_{(i,i+1)} : \mathcal{R} \to \mathcal{R}, i = 1,\ldots,n-1$, and $\mu_0 : \mathcal{R} \to \mathcal{R}$ are known continuous functions of their arguments. $\phi_i : \mathcal{R}^i \to \mathcal{R}, i = 1,\ldots,n$, are uncertain continuous functions[2] which can contain both functional and parametric uncertainties. The bounds assumed on $\phi_i, i = 1,\ldots,n$, (Assumption A2) allow cross-products of unknown parameters and unmeasured states with no magnitude bound or sign information on the unknown parameters being required. While earlier control design techniques such as the classical high-gain designs [1–4] and backstepping [5] cannot handle cross-products of unknown parameters and unmeasured states, the dynamic scaling-based dual high-gain observer/controller design approach developed in our recent papers [6,7] provides a flexible framework which can accomodate such cross-products.

High gain as a technique for controller and observer designs has been investigated extensively in the literature. The well-known adaptive high-gain controller given in its basic form by $u = -ry, \dot{r} = y^2$ is applicable to minimum-phase systems with relative-degree one [1,4]. Static high-gain scaling based observers [2,3] which introduce observer gains $r,\ldots,r^n$ with a constant r provide semiglobal solutions. In [8], a high-gain observer and a backstepping controller were designed for systems of form (1) with $\phi_{(i,i+1)} = 1, i = 1,\ldots,n-1$, and with $\phi_i, i = 1,\ldots,n$, being known functions of $x_1,\ldots,x_i$ incrementally linear in unmeasured states in the sense that $|\phi_i(x_1,\ldots,x_i) - \phi_i(x_1,\hat{x}_2,\ldots,\hat{x}_i)| \leq \Gamma(x_1)\sum_{j=2}^{i}|\hat{x}_j - x_j|$ with $\Gamma(x_1)$ being a known function.

A dual high-gain observer/controller design approach was introduced in [6,7] based on the solution of a pair of coupled Lyapunov inequalities which were shown to be always solvable under a cascading dominance assumption on the upper diagonal terms $\phi_{(i,i+1)}$ [9,7] which is closely linked to the Cascading Upper Diagonal Dominance (CUDD) condition introduced in [10]. In [7], the functions $\phi_i, i = 1,\ldots,n$, were allowed to contain functional and parametric uncertainties coupled with all the states. It was seen that a complexity of bounds on the uncertain terms ϕ_i does not result in complexity of the controller, observer, or Lyapunov function, but is instead handled through the dynamics of the high-gain scaling. However, [7] required a magnitude bound on the uncertain parameters in the system. The requirement of a magnitude bound on unknown parameters was removed in [11] using a time-varying dynamics of the high-gain scaling parameter with the basic idea being to asymptotically (as $t \to \infty$) guarantee sufficient gain to dominate the unknown parameters while retaining closed-loop stability. This provided the first output-feedback globally asymptotically stabilizing solution to the following benchmark open problem proposed in our earlier papers [12,7]

$$\dot{x}_1 = x_2\ ;\ \dot{x}_2 = x_3\ ;\ \dot{x}_3 = u + \theta_0 x_1^2 x_3 \tag{2}$$

with u being the input, $y = x_1$ the output, and θ_0 an uncertain parameter of unknown sign and with no available magnitude bounds. System (2) is of a very simple form with a single nonlinearity and a single unknown parameter. If any of the components of $\theta_0 x_1^2 x_3$ are dropped, the solution can be obtained using available techniques. If θ_0 is known, [13] and [9] provide controllers of dynamic orders 9 and 3, respectively. If x_1^2 is removed, the system is linear. If x_3 is removed, the system is in standard output-feedback canonical form [14,5]. If a magnitude bound on θ_0 is available, a solution is provided by [7]. However, with θ_0 completely unknown, no output-feedback control design technique prior to [11] can globally asymptotically stabilize the system.

A time-invariant dynamic controller based on a factorization of the scaling parameter r into two dynamic scaling parameters as $r = LM$ was recently introduced in [15] for a subclass of systems of form (1) with all the upper diagonal terms $\phi_{(i,i+1)}, i = 1,\ldots,n-1$, required to be identically equal to 1 and with the output dependence of the unknown functions $\phi_i, i = 1,\ldots,n$, required to be polynomially bounded. In this paper, we develop our control design technique of [11] further and show that the time varying component of the scaling parameter dynamics can be eliminated *without* requiring the restrictions on $\phi_{(i,i+1)}, i = 1,\ldots,n-1$, and $\phi_i, i = 1,\ldots,n$, introduced in [15]. The

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[1] The set of real numbers $(-\infty,\infty)$, the set of nonnegative real numbers $[0,\infty)$, and the set of real k-dimensional column vectors are denoted by $\mathcal{R}$, $\mathcal{R}^+$, and $\mathcal{R}^k$, respectively.

[2] The functions $\phi_i, i = 1,\ldots,n$, can be time-varying and can depend on all the states and the input. However, they are shown in (1) to only depend on subsets of the state to emphasize the triangular structure of the state dependence of the bounds to be introduced in Assumption A2.

main design highlights include a novel design of the observer featuring a $\frac{\dot{r}}{r}$ term and a new form of the dynamics of the adaptation parameter and the scaling parameter incorporating an appropriate dependence on the observer error of the first state. The proposed design technique provides an autonomous dynamic controller for systems of form (1) with the full generality of [11] in terms of assumptions on system terms. The required assumptions and the statement of the main result of the paper are provided in Section II. The observer and controller designs are presented in Section III and the stability analysis is contained in Section IV. The design for system (2) is illustrated in Section V. Extension of the design to systems with ISS appended dynamics and inverse dynamics is briefly outlined in Section VI.

II. Assumptions and Statement of Main Result

Assumption A1: System (1) is observable and controllable, i.e., a constant $\sigma > 0$ exists such that for all $x_1 \in \mathcal{R}$,

$$|\phi_{(i,i+1)}(x_1)| \geq \sigma\ ,\ 1 \leq i \leq n-1\ ;\ |\mu_0(x_1)| \geq \sigma. \quad (3)$$

Assumption A2: A continuous function $\Gamma : \mathcal{R} \to \mathcal{R}^+$ is known such that

$$|\phi_i(x_1,\ldots,x_i)| \leq \theta\Gamma(x_1)\sum_{j=1}^{i}|x_j|\ ,\ 1 \leq i \leq n \quad (4)$$

for all $x \in \mathcal{R}^n$ with $\theta \geq 0$ being an unknown parameter (with no knowledge of magnitude bounds required).

Assumption A3: Positive constants $\overline{\rho}_i$ and $\underline{\rho}_i$ exist such that for all $x_1 \in \mathcal{R}$

$$|\phi_{(i,i+1)}(x_1)| \geq \overline{\rho}_i|\phi_{(i-1,i)}(x_1)|, i = 2,\ldots,n-1 \quad (5)$$

$$|\phi_{(i,i+1)}(x_1)| \leq \underline{\rho}_i|\phi_{(i-1,i)}(x_1)|, i = 2,\ldots,n-1. \quad (6)$$

Remark 1: Assumptions A1-A3 are weaker than the set of assumptions required in both [11] and [15]. In [11], the unknown functions ϕ_i were required to satisfy the bound[3] $|\phi_i| \leq [\Gamma_0(x_1) + \theta\Gamma_1(x_1)]\sum_{j=1}^{i}|x_j|$ with $\Gamma_1(x_1)$ required to be $O[s]$ around the origin, i.e., $\Gamma_1(x_1) \leq |x_1|\overline{\Gamma}_1(x_1)$. In [15], $\Gamma(x_1)$ was required to be polynomially bounded, i.e., $\Gamma(x_1) \leq p_0 + p_1|x_1|^k$ with p_0, p_1, and k being positive constants. Furthermore, in [15], it was required that the terms $\phi_{(i,i+1)}, i = 1,\ldots,n-1$, must all be identically equal to 1.

Remark 2: Assumption A3 requires ratios of the "upper-diagonal" terms $\phi_{(i,i+1)}$ to be bounded. The condition (5) requires the upper-diagonal terms closer to the input to be larger (in a nonlinear function sense) while condition (6) requires the upper-diagonal terms closer to the output to be larger. The conditions (5) and (6) constitute the cascading dominance assumptions [10] in the controller context and observer context, respectively, and are related to uniform solvability of coupled Lyapunov inequalities [9,7] which are instrumental in the design of controller and observer gains in a dual dynamic high-gain design. Using Theorems A1 and A2 in [7], the conditions on $\phi_{(i,i+1)}$ in Assumptions A1 and A3 are necessary and sufficient for the existence of functions $g_1(x_1),\ldots,g_n(x_1), k_2(x_1),\ldots,k_n(x_1)$, symmetric positive-definite matrices P_o and P_c, and positive constants ν_o, $\overline{\nu}_o$, $\underline{\nu}_o$, ν_c, $\overline{\nu}_c$, and $\underline{\nu}_c$ to satisfy for all $x_1 \in \mathcal{R}$

$$\left.\begin{array}{c} P_oA_o(x_1) + A_o^T(x_1)P_o \leq -\nu_o|\phi_{(1,2)}(x_1)|I_n \\ \underline{\nu}_oI_n \leq P_o(D_o - \frac{1}{2}I_n) + (D_o - \frac{1}{2}I_n)P_o \leq \overline{\nu}_oI_n \end{array}\right\} \quad (7)$$

$$\left.\begin{array}{c} P_cA_c(x_1) + A_c^T(x_1)P_c \leq -\nu_c|\phi_{(2,3)}(x_1)|I_{n-1} \\ \underline{\nu}_cI_{n-1} \leq P_c(D_c - \frac{1}{2}I_{n-1}) + (D_c - \frac{1}{2}I_{n-1})P_c \leq \overline{\nu}_cI_{n-1} \end{array}\right\} \quad (8)$$

[3]For notational convenience, we drop the arguments of functions when no confusion will result.

where

$$A_o = \begin{bmatrix} -g_1 & \phi_{(1,2)} & 0 & 0 & \ldots \\ -g_2 & 0 & \phi_{(2,3)} & 0 & \ldots \\ \vdots & & & \ddots & \\ -g_{n-1} & & & & \phi_{(n-1,n)} \\ -g_n & 0 & & \ldots & 0 \end{bmatrix} \quad (9)$$

$$A_c = \begin{bmatrix} 0 & \phi_{(2,3)} & 0 & 0 & \ldots \\ 0 & 0 & \phi_{(3,4)} & 0 & \ldots \\ \vdots & & & \ddots & \\ 0 & & & & \phi_{(n-1,n)} \\ -k_2 & -k_3 & & \ldots & -k_n \end{bmatrix} \quad (10)$$

$$C = [1,0,\ldots,0] \quad (11)$$

$$D_o = \text{diag}(1,1,2,3,\ldots,n-1) \quad (12)$$

$$D_c = \text{diag}(1,2,3,\ldots,n-1), \quad (13)$$

I_k denotes an identity matrix of dimension $k \times k$, and $\text{diag}(a_1,\ldots,a_k)$ denotes the $k \times k$ diagonal matrix with the i^{th} diagonal element being a_i.

Furthermore, by Theorem A1 in [7], $g_1(x_1),\ldots,g_n(x_1)$ can be picked to be linear constant-coefficient combinations of $\phi_{(1,2)}(x_1),\ldots,\phi_{(n-1,n)}(x_1)$. Hence, using Assumption A3, a positive constant $\overline{G}$ exists such that $\sqrt{\sum_{i=1}^{n}g_i^2(x_1)} \leq \overline{G}|\phi_{(1,2)}(x_1)|$.

Theorem 1: Under Assumptions A1-A3, positive constants a and b and continuous functions $\zeta : \mathcal{R}^2 \to \mathcal{R}$, $\Theta_1 : \mathcal{R} \to \mathcal{R}^+$, $\Theta_2 : \mathcal{R}^2 \to \mathcal{R}^+$, $\gamma : \mathcal{R}^3 \to \mathcal{R}^+$, $g_i : \mathcal{R} \to \mathcal{R}, i = 1,\ldots,n$, and $k_i : \mathcal{R} \to \mathcal{R}, i = 2,\ldots,n$ can be chosen such that all solution trajectories of the closed-loop system formed by the dynamic controller given by

$$\left.\begin{array}{l} \dot{\hat{x}}_1 = \phi_{(1,2)}(x_1)\hat{x}_2 - \frac{\dot{r}}{r}(\hat{x}_1 - x_1) - rg_1(x_1)[\hat{x}_1 - x_1] \\ \dot{\hat{x}}_i = \phi_{(i,i+1)}(x_1)\hat{x}_{i+1} \\ \quad\quad -r^ig_i(x_1)[\hat{x}_1 - x_1]\ ,\ i = 2,\ldots,n-1 \\ \dot{\hat{x}}_n = \mu_0(x_1)u - r^ng_n(x_1)[\hat{x}_1 - x_1] \end{array}\right\} \quad (14)$$

$$u = -\frac{r^n}{\mu_0}\sum_{i=2}^{n}k_i(x_1)\eta_i \quad (15)$$

$$\eta_2 = \frac{\hat{x}_2 + \zeta(x_1,\hat{\theta})}{r}\ ;\ \eta_i = \frac{\hat{x}_i}{r^{i-1}}\ ,\ i = 3,\ldots,n \quad (16)$$

$$\dot{r} = r[-a(r-1) + b\gamma(x_1,\hat{x}_1,\hat{\theta})]\ ;\ r(0) \geq 1 \quad (17)$$

$$\dot{\hat{\theta}} = \Theta_1(x_1)x_1^2 + \Theta_2(x_1,r)(\hat{x}_1 - x_1)^2\ ;\ \hat{\theta}(0) > 0, \quad (18)$$

in closed loop with system (1) starting from any initial condition $(x(0),\hat{x}(0),r(0),\hat{\theta}(0)) \in \mathcal{R}^n \times \mathcal{R}^n \times [1,\infty) \times (0,\infty)$ where $\hat{x} = [\hat{x}_1,\ldots,\hat{x}_n]^T$ have the following properties:

- Solution trajectories exist on the time interval $[0,\infty)$ and all closed-loop signals are bounded on $[0,\infty)$.
- The states $x_1,\ldots,x_n$, the observer errors $e_i \stackrel{\triangle}{=} \hat{x}_i - x_i, i = 1,\ldots,n$, and the control input u asymptotically converge to zero as $t \to \infty$.

III. Observer and Controller

Observer: A full-order observer for system (1) is given by (14) where r is the dynamic high-gain scaling parameter and $g_1,\ldots,g_n$ are functions chosen as in Remark 2. The observer errors e_i and scaled observer errors ϵ_i are defined as

$$e_i = \hat{x}_i - x_i\ ;\ \epsilon_i = \frac{e_i}{r^{i-1}}\ ,\ 1 \leq i \leq n. \quad (19)$$

From (19), $\epsilon_1 = e_1$. The dynamics of the scaled observer error vector $\epsilon = [\epsilon_1, \ldots, \epsilon_n]^T$ are given by

$$\dot{\epsilon} = rA_o\epsilon - \frac{\dot{r}}{r}D_o\epsilon - \overline{\Phi} \quad (20)$$

$$\overline{\Phi} = [\overline{\Phi}_1, \ldots, \overline{\Phi}_n]^T \ , \ \overline{\Phi}_i = \frac{\phi_i}{r^{i-1}} \quad (21)$$

where A_o and D_o are defined in (9) and (12), respectively. The term $-\frac{\dot{r}}{r}e_1$ introduced in the dynamics of $\hat{x}_1$ in (14) contributes a value of 1 to the $(1,1)$ element of D_o thus ensuring the positive-definiteness of the matrix $(D_o - \frac{1}{2}I_n)$. This is crucial to the solvability of the second Lyapunov inequality in (7).

Controller: The control law is given by (15) where the controller gain functions $k_2, \ldots, k_n$ are chosen as in Remark 2 and $\eta_2, \ldots, \eta_n$ are given by (16). The design function ζ is picked to be of the form

$$\zeta(x_1, \hat{\theta}) = (1+\hat{\theta})x_1\zeta_1(x_1) \quad (22)$$

with ζ_1 being a continuously differentiable function and $\hat{\theta}$ a parameter estimator. The signals η_i, $i = 2, \ldots, n$, are scaled observer estimates of the states x_i with an additional design freedom ζ incorporated into η_2. The dynamics of $\eta = [\eta_2, \ldots, \eta_n]^T$ are

$$\dot{\eta} = rA_c\eta - \frac{\dot{r}}{r}D_c\eta - rG_2\epsilon_1 + H(\eta_2 - \epsilon_2) + \Xi \quad (23)$$

with A_c and D_c defined in (10) and (13), respectively, and

$$G_2 = [g_2, \ldots, g_n]^T \quad (24)$$

$$H = \left[(1+\hat{\theta})\left\{\zeta_1' x_1 + \zeta_1\right\}\phi_{(1,2)}, 0, \ldots, 0\right]^T \quad (25)$$

$$\Xi = \frac{1}{r}\left[\dot{\hat{\theta}}\zeta_1 x_1 + (1+\hat{\theta})\left\{\zeta_1' x_1 + \zeta_1\right\} \times\left\{\phi_1 - (1+\hat{\theta})\zeta_1 x_1 \phi_{(1,2)}\right\}, 0, \ldots, 0\right]^T \quad (26)$$

where $\zeta_1'(x_1)$ denotes the partial derivative evaluated at x_1 of ζ_1 with respect to its argument.

IV. Stability Analysis and Proof of Theorem 1

To analyze closed-loop stability, the observer and controller Lyapunov functions are defined as

$$V_o = r\epsilon^T P_o \epsilon \ , \ V_c = r\eta^T P_c \eta + \frac{\beta_1}{2}x_1^2 \quad (27)$$

where β_1 is a positive constant free to be picked by the designer. Differentiating V_o and V_c and using (20) and (23),

$$\begin{aligned} \dot{V}_o &= r^2\epsilon^T[P_oA_o + A_o^TP_o]\epsilon - 2r\epsilon^TP_o\overline{\Phi} \\ &\quad -\dot{r}\epsilon^T\left[P_o(D_o - \frac{1}{2}I_n) + (D_o - \frac{1}{2}I_n)P_o\right]\epsilon \quad (28) \\ \dot{V}_c &= r^2\eta^T[P_cA_c + A_c^TP_c]\eta \\ &\quad -\dot{r}\eta^T\left[P_c(D_c - \frac{1}{2}I_{n-1}) + (D_c - \frac{1}{2}I_{n-1})P_c\right]\eta \\ &\quad -2r^2\eta^TP_cG_2\epsilon_1 + 2r\eta^TP_cH(\eta_2 - \epsilon_2) + 2r\eta^TP_c\Xi \\ &\quad +\beta_1x_1[\phi_1 + (r\eta_2 - r\epsilon_2 - \zeta)\phi_{(1,2)}]. \quad (29) \end{aligned}$$

Using Assumption A2 and the properties $r(t) \geq 1$ and $\hat{\theta}(t) > 0$ as can be seen from (17) and (18), bounds on various terms appearing in (28) and (29) can be obtained as follows:

$$\begin{aligned} -2r\epsilon^TP_o\overline{\Phi} &\leq 3r\lambda_{max}(P_o)n\theta\Gamma\left[|\epsilon|^2 + |\eta|^2\right] \\ &\quad + \frac{c}{\beta_2}\lambda_{max}^2(P_o)[1+(1+\hat{\theta})|\zeta_1|]^2n\theta^2\Gamma^2|\epsilon|^2 \\ &\quad + \frac{\beta_2}{c}x_1^2 + \frac{\nu_o}{4}r^2|\phi_{(1,2)}||\epsilon|^2 \\ &\quad + \frac{4}{\sigma\nu_o}\lambda_{max}^2(P_o)\theta^2\Gamma^2x_1^2 \quad (30) \end{aligned}$$

$$\begin{aligned} -2r^2\eta^TP_cG_2\epsilon_1 &\leq \frac{\nu_c}{4}r^2|\phi_{(2,3)}||\eta|^2 \\ &\quad + \frac{4}{\nu_c}r^2\lambda_{max}^2(P_c)\overline{G}^2\frac{|\phi_{(1,2)}|^2}{|\phi_{(2,3)}|}\epsilon_1^2 \quad (31) \end{aligned}$$

$$\begin{aligned} 2r\eta^TP_cH(\eta_2 - \epsilon_2) &\leq 2r\lambda_{max}(P_c)(1+\hat{\theta}) \\ &\quad \times|\zeta_1'x_1 + \zeta_1||\phi_{(1,2)}||\eta|^2 \\ &\quad + \frac{4}{c\nu_o}r\lambda_{max}^2(P_c)(1+\hat{\theta})^2 \\ &\quad \times|\zeta_1'x_1 + \zeta_1|^2|\phi_{(1,2)}||\eta|^2 \\ &\quad + \frac{c\nu_o}{4}r^2|\phi_{(1,2)}|\epsilon_2^2 \quad (32) \end{aligned}$$

$$\begin{aligned} 2r\eta^TP_c\Xi &\leq \beta_2x_1^2 + \frac{3}{\beta_2}r\lambda_{max}^2(P_c)\left[\dot{\hat{\theta}}^2\zeta_1^2 \right. \\ &\quad +(1+\hat{\theta})^4(\zeta_1'x_1 + \zeta_1)^2\zeta_1^2\phi_{(1,2)}^2 \\ &\quad \left. +(1+\hat{\theta})^2(\zeta_1'x_1 + \zeta_1)^2\theta^2\Gamma^2\right]|\eta|^2 \quad (33) \end{aligned}$$

$$\beta_1x_1\phi_1 \leq \beta_1\theta\Gamma x_1^2 \quad (34)$$

$$\begin{aligned} \beta_1x_1\phi_{(1,2)}[r\eta_2 - r\epsilon_2] &\leq \frac{\nu_c}{4}r^2|\phi_{(2,3)}|\eta_2^2 + \frac{c\nu_o}{4}r^2|\phi_{(1,2)}|\epsilon_2^2 \\ &\quad + \frac{\beta_1^2}{\nu_c}\frac{\phi_{(1,2)}^2}{|\phi_{(2,3)}|}x_1^2 + \frac{\beta_1^2}{c\nu_o}|\phi_{(1,2)}|x_1^2 \quad (35) \end{aligned}$$

where β_2, β_3, and c are design parameters which will be picked to be positive constants and $\lambda_{max}(P)$ denotes the maximum eigenvalue of a symmetric positive-definite matrix P.

Closed-loop stability is analyzed using the Lyapunov function

$$V = cV_o + V_c = cr\epsilon^TP_o\epsilon + r\eta^TP_c\eta + \frac{\beta_1}{2}x_1^2. \quad (36)$$

Using (7), (8), and (28)-(35), and picking c to be any positive constant such that $c \geq \frac{8}{\nu_o\nu_c\underline{\rho}_2}\lambda_{max}^2(P_c)\overline{G}^2$,

$$\begin{aligned} \dot{V} &\leq -\frac{c\nu_o}{4}r^2|\phi_{(1,2)}||\epsilon|^2 - \frac{\nu_c}{2}r^2|\phi_{(2,3)}||\eta|^2 - \beta_1x_1\phi_{(1,2)}\zeta \\ &\quad +[q_1(x_1) + \theta^*q_2(x_1)]x_1^2 \\ &\quad -c\dot{r}\epsilon^T\left[P_o(D_o - \frac{1}{2}I_n) + (D_o - \frac{1}{2}I_n)P_o\right]\epsilon \\ &\quad -\dot{r}\eta^T\left[P_c(D_c - \frac{1}{2}I_{n-1}) + (D_c - \frac{1}{2}I_{n-1})P_c\right]\eta \\ &\quad +r[w_1(x_1, \hat{\theta}, \dot{\hat{\theta}}) + \theta^*w_2(x_1, \hat{\theta})]\{|\epsilon|^2 + |\eta|^2\} \quad (37) \end{aligned}$$

where $\theta^* \triangleq \max\{1, \theta + \theta^2\}$ and

$$q_1(x_1) = 2\beta_2 + \frac{\beta_1^2}{\nu_c}\frac{\phi_{(1,2)}^2}{|\phi_{(2,3)}|} + \frac{\beta_1^2}{c\nu_o}|\phi_{(1,2)}| \quad (38)$$

$$q_2(x_1) = \beta_1\Gamma + \frac{4c}{\sigma\nu_o}\lambda_{max}^2(P_o)\Gamma^2 \quad (39)$$

$$w_1(x_1, \hat{\theta}, \dot{\hat{\theta}}) = 2\lambda_{max}(P_c)(1+\hat{\theta})|\zeta_1'x_1 + \zeta_1||\phi_{(1,2)}|$$

$$+\frac{4}{c\nu_o}\lambda_{max}^2(P_c)(1+\hat{\theta})^2|\zeta_1' x_1+\zeta_1|^2|\phi_{(1,2)}|$$
$$+\frac{3}{\beta_2}\lambda_{max}^2(P_c)\Big[\dot{\hat{\theta}}^2\zeta_1^2$$
$$+(1+\hat{\theta})^4(\zeta_1' x_1+\zeta_1)^2\zeta_1^2\phi_{(1,2)}^2\Big] \quad (40)$$

$$w_2(x_1,\hat{\theta}) \;=\; 3c\lambda_{max}(P_o)n\Gamma + \frac{1}{\beta_2}c^2\lambda_{max}^2(P_o)n\Gamma^2[1+(1+\hat{\theta})|\zeta_1|]^2 + \frac{3}{\beta_2}\lambda_{max}^2(P_c)(1+\hat{\theta})^2(\zeta_1' x_1+\zeta_1)^2\Gamma^2. \quad (41)$$

Picking b to be an arbitrary positive constant, choose $a > 0$ small enough to ensure that

$$\max(-\frac{\sigma c\nu_o}{4}+ac\overline{\nu}_o, -\frac{\sigma\nu_c}{2}+a\overline{\nu}_c) \;=\; -a^* < 0, \quad (42)$$

and choose $\zeta_1(x_1)$ such that

$$-\beta_1\zeta_1(x_1)\phi_{(1,2)}(x_1)+q_1(x_1)+q_2(x_1) \;\le\; -\zeta_1^*(x_1) \quad (43)$$

with ζ_1^* being a positive function of x_1 bounded below by a positive constant $\underline{\zeta}_1^*$. The parameter estimator dynamics are chosen as shown in (18) with

$$\Theta_1(x_1) \;=\; \frac{1}{\beta_\theta}q_2(x_1) \;\;,\;\; \Theta_2(x_1,r) = \frac{1}{\beta_\theta}r^2\phi_{(1,2)}^2 \quad (44)$$

where $\beta_\theta > 0$ is a design parameter. Note that the parameter estimate $\hat{\theta}(t)$ with dynamics (18) is a monotonically nondecreasing function of time. The design function γ is picked to be

$$\gamma(x_1,\hat{x}_1,\hat{\theta}) \;=\; \frac{1}{b\min(c\underline{\nu}_o,\underline{\nu}_c)} \times\Bigg[w_1\Big(x_1,\hat{\theta},\Theta_1(x_1)x_1^2+\Theta_2(x_1,r)(\hat{x}_1-x_1)^2\Big) + \frac{1}{\beta_3}w_2^2(x_1,\hat{\theta})(1+x_1^2)+\hat{\theta}+\beta_4\Bigg] \quad (45)$$

with β_4 being a design parameter which can be picked to be any positive constant. Using (17), (42), (43) and (45), (37) reduces to

$$\dot{V} \;\le\; -a^*r^2[|\epsilon|^2+||\eta|^2]-x_1^2\zeta_1^*(x_1)+(\theta^*-\hat{\theta})q_2(x_1)x_1^2 + r\Big\{\theta^* w_2(x_1,\hat{\theta})-\frac{1}{\beta_3}w_2^2(x_1,\hat{\theta})(1+x_1^2)-\hat{\theta}-\beta_4\Big\} \times[|\epsilon|^2+|\eta|^2]. \quad (46)$$

The *parameter estimation error* is defined to be $(\hat{\theta}-\overline{\theta})$ with $\overline{\theta} \stackrel{\triangle}{=} \max\{\theta^*, \frac{\beta_3}{4}\theta^{*^2}\}$. Note that $\overline{\theta} \ge 1$ since θ^* was defined as $\max\{1,\theta+\theta^2\}$. A new Lyapunov function is defined including a quadratic of the parameter estimation error $(\hat{\theta}-\overline{\theta})$ as

$$\overline{V} \;=\; V+\frac{\beta_\theta}{2}(\hat{\theta}-\overline{\theta})^2. \quad (47)$$

Using (44) and (46),

$$\dot{\overline{V}} \;\le\; -a^*r^2[|\epsilon|^2+||\eta|^2]-x_1^2\zeta_1^*(x_1) +(\hat{\theta}-\overline{\theta})r^2\phi_{(1,2)}^2[\hat{x}_1-x_1]^2 + r\Big\{\theta^* w_2(x_1,\hat{\theta})-\frac{1}{\beta_3}w_2^2(x_1,\hat{\theta})(1+x_1^2)-\hat{\theta}-\beta_4\Big\} \times[|\epsilon|^2+|\eta|^2]. \quad (48)$$

Closed-loop stability is proved through a sequence of lemmas below. Local existence of solutions is guaranteed by the assumptions on the functions ϕ_i and $\phi_{(i,i+1)}$. Let the maximal interval of existence of solutions be $[0,t_f)$. The proof of Theorem 1 utilizes Lemmas 1-4 to infer that $t_f = \infty$ (i.e., solutions exist for all time) and that in the limit as $t\to\infty$, the states $x_1,\ldots,x_n$, the observer errors $e_1,\ldots,e_n$, and the control input u converge to zero. The proofs of Lemmas 1-4 are given in the Appendix.

Lemma 1: If $\sup_{t\in[0,t_f)}V(t) < \infty$ and $\sup_{t\in[0,t_f)}\hat{\theta}(t) < \infty$, then $\sup_{t\in[0,t_f)}r(t) < \infty$.

Lemma 2: If $\sup_{t\in[0,t_f)}\hat{\theta}(t) > \overline{\theta}$, then $t_f = \infty$, $\lim_{t\to\infty}V(t) = 0$, $\int_0^\infty V(t)dt < \infty$, and $\sup_{t\in[0,\infty)}\hat{\theta}(t) < \infty$.

Lemma 3: If $\sup_{t\in[0,t_f)}\hat{\theta}(t) \le \overline{\theta}$, then $\sup_{t\in[0,t_f)}V(t) < \infty$, and $\int_0^{t_f}V(t) < \infty$.

Lemma 4: If $\sup_{t\in[0,t_f)}\hat{\theta}(t) \le \overline{\theta}$, then $t_f = \infty$ and $\lim_{t\to\infty}V(t) = 0$.

Proof of Theorem 1: With the maximal interval of existence of solutions denoted by $[0,t_f)$, one of the following possibilities should hold: Case $\mathcal{A}_1$: $\sup_{t\in[0,t_f)}\hat{\theta}(t) \le \overline{\theta}$; Case $\mathcal{A}_2$: $\sup_{t\in[0,t_f)}\hat{\theta}(t) > \overline{\theta}$. If Case $\mathcal{A}_2$ holds, then Lemma 2 guarantees that $t_f = \infty$. On the other hand, under Case $\mathcal{A}_1$, Lemma 4 implies that $t_f = \infty$. Hence, the possibility of finite escape time is ruled out, i.e., $t_f = \infty$. Furthermore, from Lemmas 2-4, it is seen that $\sup_{t\in[0,\infty)}V(t) < \infty$, $\int_0^\infty V(t)dt < \infty$, $\sup_{t\in[0,\infty)}\hat{\theta}(t) < \infty$, and $\lim_{t\to\infty}V(t) = 0$. As shown in the proofs of Lemmas 1 and 2, the boundedness of all closed-loop signals on the time interval $[0,\infty)$ follows from the boundedness of $\hat{\theta}(t)$, $V(t)$, and $\int_0^t V(\tau)d\tau$ on $t\in[0,\infty)$. The asymptotic convergence of $V(t)$ to zero as $t\to\infty$ implies the asymptotic convergence of the system states $x_1,\ldots,x_n$, the observer states $\hat{x}_1,\ldots,\hat{x}_n$, and the control input u to zero as $t\to\infty$, thus completing the proof of Theorem 1. $\diamond$

V. Solution to the Benchmark Problem [12,7]

The system (2) belongs to the class of systems (1) with $n = 3$, $\phi_1 = \phi_2 = 0$, $\phi_3 = \theta_0 x_1^2 x_3$, and $\phi_{(1,2)} = \phi_{(2,3)} = \mu_0 = 1$. It is easily seen that Assumptions A1-A3 are satisfied with $\sigma = 1$, $\Gamma(x_1) = x_1^2$, $\theta = |\theta_0|$, and $\overline{\rho}_2 = \underline{\rho}_2 = 1$. The inequalities (7) and (8) are satisfied with $g_1 = 6, g_2 = 13, g_3 = 5$, $k_2 = 2.9, k_3 = 3.6$, $\nu_o = 1$, $\nu_c = 5.6608$, $\overline{\nu}_o = 5.9897$, $\underline{\nu}_o = 0.017912$, $\overline{\nu}_c = 7.5428$, $\underline{\nu}_c = 2.3572$, and

$$P_o \;=\; \begin{bmatrix} 1.6233 & -0.5 & -0.54795 \\ -0.5 & 0.54795 & -0.5 \\ -0.54795 & -0.5 & 1.8575 \end{bmatrix} \quad (49)$$

$$P_c \;=\; \begin{bmatrix} 6.6 & 1 \\ 1 & 1.1 \end{bmatrix}. \quad (50)$$

The design procedure in Sections III and IV yields the following dynamic output-feedback controller

$$\dot{\hat{x}}_1 \;=\; \hat{x}_2-\frac{\dot{r}}{r}(\hat{x}_1-x_1)-6r(\hat{x}_1-x_1)$$
$$\dot{\hat{x}}_2 \;=\; \hat{x}_3-13r^2(\hat{x}_1-x_1)$$
$$\dot{\hat{x}}_3 \;=\; u-5r^3(\hat{x}_1-x_1)$$
$$u \;=\; -r^3(2.9\eta_2+3.6\eta_3)$$
$$\eta_2 \;=\; \frac{\hat{x}_2+0.7(1+\hat{\theta})x_1}{r} \;\;;\;\; \eta_3 \;=\; \frac{\hat{x}_3}{r^2}$$
$$\gamma(x_1,\hat{x}_1,\hat{\theta}) \;=\; 0.1439(1+\hat{\theta})+0.00029(1+\hat{\theta})^2$$

$$
\begin{aligned}
& +0.0058(1+\hat{\theta})^4+0.0086\dot{\hat{\theta}}^2+0.1439x_1^4 \\
& +0.1439x_1^8(1+\hat{\theta})^4+0.02878 \\
\dot{r} &= r[-0.0396(r-1)+\gamma(x_1,\hat{x}_1,\hat{\theta})] \\
\dot{\hat{\theta}} &= 0.1x_1^2+r^2(\hat{x}_1-x_1)^2 \qquad (51)
\end{aligned}
$$

which guarantees boundedness of all closed-loop signals on the time interval $[0,\infty)$ and asymptotic convergence of the signals $x_1, x_2, x_3, \hat{x}_1, \hat{x}_2, \hat{x}_3$, and u to zero as $t \to \infty$. Simulation results with the initial conditions $x_1 = x_2 = x_3 = 1$, $\hat{x}_1 = 1$, $\hat{x}_2 = \hat{x}_3 = 0$, $\hat{\theta} = 0.2$, and $r = 5$ are shown in Figure 1 for system (2) with $\theta_0 = 2$.

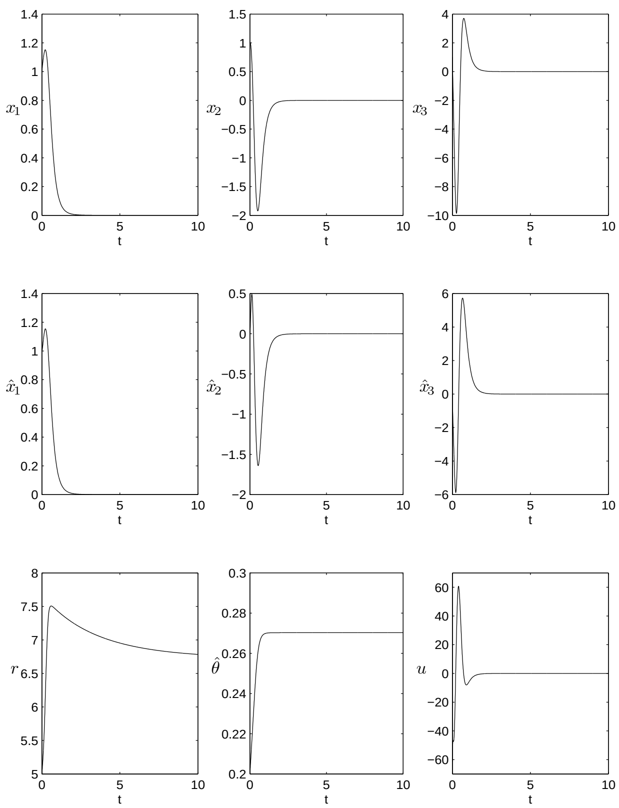

Fig. 1. Simulation results for system (2) with $\theta_0 = 2$.

VI. Extension to Systems with ISS Appended Dynamics

By combining the high-gain scaling dynamics design technique in this paper with the technique for handling ISS appended dynamics driven by all states in [7], the proposed results can be extended to the more general class of systems

$$
\begin{aligned}
\dot{z}_i &= q_i(z,x,t)\ ,\ i=1,\ldots,n \\
\dot{x}_i &= \phi_i(t,z,x,u)+\phi_{(i,i+1)}(x_1)x_{i+1} \\
& \quad +\psi_i(z,x,t)\ \ ,\ \ i=1,\ldots,s-1 \\
\dot{x}_i &= \phi_i(t,z,x,u)+\phi_{(i,i+1)}(x_1)x_{i+1}+\mu_{i-s}(x_1)u \\
& \quad +\psi_i(z,x,t)\ \ ,\ \ i=s,\ldots,n \\
y &= x_1 \qquad (52)
\end{aligned}
$$

where $z_i \in \mathcal{R}^{n_{z_i}}$ are the (unmeasurable) states of appended Input-to-State Stable (ISS) dynamics[16] and $z = [z_1^T, \ldots, z_n^T]^T$. s is the relative degree of the system and $[x_{s+1}, \ldots, x_n]^T$ is the state of the inverse dynamics. The appended dynamics are driven by all the system states with a triangular structure of ISS interconnections, i.e., z_i is allowed to have nonzero nonlinear gains from states $x_1, \ldots, x_i$. The uncertain functions ϕ_i are required to be bounded by the product of an uncertain parameter θ, a nonlinear function $\Gamma(x_1)$, and a linear combination of $|x_1|, \ldots, |x_i|, |z_2|, \ldots, |z_i|$, and a nonlinear function of $|z_1|$. The inverse dynamics subsystem is assumed to be ISS with nonzero nonlinear gains from $x_1, \ldots, x_s, z_1, \ldots, z_s$. While previous techniques [17] required ISS dynamics and inverse dynamics to be driven only by x_1, [7] provided a method using the dynamic high-gain scaling approach to handle ISS appended dynamics and inverse dynamics driven by all the system states. The design in [7] utilized dynamics of the high-gain scaling parameter of the form $\dot{r} = \lambda(R(x_1, \hat{\theta}, \dot{\hat{\theta}}) - r)\Omega(r, x_1, \hat{\theta}, \dot{\hat{\theta}})$ with R, λ, and Ω being suitably chosen functions. The Lyapunov function in [7] incorporates appropriately scaled versions of the ISS Lyapunov functions of the inverse dynamics and the appended dynamics. By using the techniques in [7] and the design of the observer, the high-gain scaling parameter dynamics, and the adaptation parameter dynamics in this paper, the proposed controller can be extended to obtain global output-feedback results for (52). The details are omitted here for brevity.

Appendix: Proofs of Lemmas 1-4

Proof of Lemma 1: The boundedness of $V(t)$ implies the boundedness of x_1 and $\hat{x}_1$. The boundedness of x_1, $\hat{x}_1$, and $\hat{\theta}$ implies the boundedness of $\gamma(x_1, \hat{x}_1, \hat{\theta})$. From the dynamics (17), the boundedness of $r(t)$ on any time interval $[0, T)$ follows from the boundedness of $\gamma(x_1, \hat{x}_1, \hat{\theta})$ on the same time interval $[0, T)$. $\diamond$

Proof of Lemma 2: From the dynamics of $\hat{\theta}$ given in (18) and (44), it is seen that $\hat{\theta}(t)$ is monotonically nondecreasing. Hence, if $\sup_{t\in[0,t_f)} \hat{\theta}(t) > \overline{\theta}$, then a time $T \in [0, t_f)$ exists such that $\hat{\theta}(t) \geq \overline{\theta}$ for all $t \in [T, t_f)$. Hence, using (46) and the inequalities $\overline{\theta} \geq \theta^*$, $\overline{\theta} \geq \frac{\beta_3}{4}{\theta^*}^2$, and

$$
\theta^* w_2 \;\leq\; \frac{\beta_3}{4}{\theta^*}^2 + \frac{1}{\beta_3} w_2^2, \qquad (53)
$$

we obtain, for all $t \in [T, t_f)$,

$$
\begin{aligned}
\dot{V} &\leq -a^* r^2[|\epsilon|^2 + ||\eta|^2] - x_1^2 \zeta_1^*(x_1) \qquad (54) \\
&\leq -\chi_V V \qquad (55)
\end{aligned}
$$

where

$$
\chi_V = \min\Big(\frac{a^*}{c\lambda_{max}(P_o)}, \frac{a^*}{\lambda_{max}(P_c)}, \frac{2\underline{\zeta}_1^*}{\beta_1}\Big). \qquad (56)
$$

Hence, $V(t)$ is bounded on the time interval $[0, t_f)$. Also, integrating both sides of (54) and (55), it is seen that $\int_0^{t_f} V(\tau)d\tau$, $\int_0^{t_f} x_1^2(\tau)d\tau$, and $\int_0^{t_f} r^2(\tau)e_1^2(\tau)d\tau$ are finite. Noting that Γ and $\phi_{(1,2)}$ are continuous functions, $\int_0^{t_f} q_2(x_1(\tau))x_1^2(\tau)d\tau$ and $\int_0^{t_f} r^2(\tau)\phi_{(1,2)}^2(x_1(\tau))e_1^2(\tau)d\tau$ are also finite. Hence, $\hat{\theta}(t)$ which is governed by the dynamics given in (18) and (44) is bounded on the time interval $[0, t_f)$. From Lemma 1, the boundedness of $r(t)$ on $[0, t_f)$ is inferred from the boundedness of $V(t)$ and $\hat{\theta}(t)$. Therefore, all closed-loop states are bounded on $[0, t_f)$. This contradicts the possibility that t_f is finite. Furthermore, from (55), it is seen that if $t_f = \infty$, then $V(t)$ asymptotically converges to zero as $t \to \infty$. $\diamond$

Proof of Lemma 3: Consider the Lyapunov functions

$$\tilde{V}_o = r\epsilon^T \tilde{P}_o \epsilon \ , \ \tilde{P}_o = \tilde{\theta} T(\tilde{\theta}) P_o T(\tilde{\theta}) \quad (57)$$

$$\tilde{V} = c\tilde{V}_o + V_c = cr\epsilon^T \tilde{P}_o \epsilon + r\eta^T P_c \eta + \frac{\beta_1}{2} x_1^2 \quad (58)$$

where $\tilde{\theta} = \max\{\overline{\theta}, \frac{4\beta_3 c}{a^*}\theta^2\}$ and $T(\tilde{\theta})$ is a diagonal matrix defined as $T(\tilde{\theta}) = \text{diag}(1, \frac{1}{\tilde{\theta}}, \frac{1}{\tilde{\theta}^2}, \ldots, \frac{1}{\tilde{\theta}^{n-1}})$. Using (7) and (17), and noting that

$$\begin{aligned} T(\tilde{\theta}) A_o T^{-1}(\tilde{\theta}) &= \tilde{\theta} A_o + [\tilde{\theta} I - T(\tilde{\theta})] GC \\ T(\tilde{\theta}) D_o T^{-1}(\tilde{\theta}) &= D_o, \end{aligned} \quad (59)$$

we obtain

$$\begin{aligned} \dot{\tilde{V}}_o \leq & -\tilde{\theta}^2 r^2 \nu_o |\phi_{(1,2)}||T(\tilde{\theta})\epsilon|^2 + a\tilde{\theta} r^2 \overline{\nu}_o |T(\tilde{\theta})\epsilon|^2 \\ & -\tilde{\theta} r(a + b\gamma)\underline{\nu}_o |T(\tilde{\theta})\epsilon|^2 \\ & +2\tilde{\theta} r^2 \epsilon^T T(\tilde{\theta}) P_o [\tilde{\theta} I_n - T(\tilde{\theta})] GCT(\tilde{\theta})\epsilon \\ & -2\tilde{\theta} r \epsilon^T T(\tilde{\theta}) P_o T(\tilde{\theta}) \overline{\Phi}. \end{aligned} \quad (60)$$

Upper bounding terms in (60) using inequalities similar to (30) and noting that $\tilde{\theta} \geq (4\beta_3 c/a^*)\theta^2$, (60) can be simplified to

$$\begin{aligned} \dot{\tilde{V}}_o \leq & -\frac{3}{4}\tilde{\theta}^2 r^2 \nu_o |\phi_{(1,2)}||T(\tilde{\theta})\epsilon|^2 + a\tilde{\theta} r^2 \overline{\nu}_o |T(\tilde{\theta})\epsilon|^2 \\ & + r^2\tilde{\theta}^2 \frac{a^*}{2c}|T(\tilde{\theta})\epsilon|^2 - \tilde{\theta} r(a+b\gamma)\underline{\nu}_o |T(\tilde{\theta})\epsilon|^2 \\ & + r^2 \tilde{\theta}^2 \frac{4c}{a^*}\phi_{(1,2)}^2 \lambda_{max}^2(P_o)\overline{G}^2 e_1^2 \\ & + \frac{9}{\beta_3}\tilde{\theta} r \lambda_{max}^2(P_o) n^2 \Gamma^2 |T(\tilde{\theta})\epsilon|^2 \\ & + 3r\lambda_{max}(P_o) n\Gamma |T_2(\tilde{\theta})\eta|^2 \\ & + \frac{c}{\beta_2}\lambda_{max}^2(P_o) n\theta^2\Gamma^2 [1 + (1+\hat{\theta})|\zeta_1|]^2 |T(\tilde{\theta})\epsilon|^2 \\ & + \frac{\beta_2}{c}x_1^2 + \frac{4}{\sigma\nu_o}\lambda_{max}^2(P_o)\theta^2\Gamma^2 x_1^2. \end{aligned} \quad (61)$$

where $T_2(\tilde{\theta}) \stackrel{\triangle}{=} \text{diag}(\frac{1}{\tilde{\theta}}, \frac{1}{\tilde{\theta}^2}, \ldots, \frac{1}{\tilde{\theta}^{n-1}})$. Using (29), (31)-(35), the definitions of a, ζ, and γ in (42), (43), and (45), respectively, and the inequalities $\theta \leq \tilde{\theta}^2$, $\epsilon_2^2 \leq \tilde{\theta}^2|T(\tilde{\theta})\epsilon|^2$, and $(\hat{\theta} - \overline{\theta}) \leq 0$ (which is the hypothesis of Lemma 3)

$$\begin{aligned} \dot{\tilde{V}} \leq & -\frac{a^*}{2} r^2[\tilde{\theta}^2 |T(\tilde{\theta})\epsilon|^2 + |\eta|^2] - x_1^2 \zeta_1^*(x_1) \\ & + r^2\tilde{\theta}^2 \frac{4c}{a^*}\phi_{(1,2)}^2 \lambda_{max}^2(P_o)\overline{G}^2 e_1^2 \\ \leq & -\chi\tilde{V} + r^2\tilde{\theta}^2 \frac{4c}{a^*}\phi_{(1,2)}^2 \lambda_{max}^2(P_o)\overline{G}^2 e_1^2 \end{aligned} \quad (62)$$

where

$$\chi = \min\Big(\frac{a^*\tilde{\theta}}{2c\lambda_{max}(P_o)}, \frac{a^*}{2\lambda_{max}(P_c)}, \frac{2\underline{\zeta}_1^*}{\beta_1}\Big). \quad (63)$$

Integrating both sides of (62) over any time interval $(0, t)$ with $t < t_f$,

$$\tilde{V}(t) - \tilde{V}(0) \leq -\chi \int_0^t \tilde{V}(\tau)d\tau + \Big[\tilde{\theta}^2 \frac{4c}{a^*}\lambda_{max}^2(P_o)\overline{G}^2\Big] \times \int_0^t r^2(\tau)\phi_{(1,2)}^2(x_1(\tau))e_1^2(\tau)d\tau. \quad (64)$$

The hypothesis of Lemma 3 states that $\hat{\theta} \leq \overline{\theta}$. From the dynamics of $\hat{\theta}$ given in (18) and (44), this implies that

$$\int_0^t r^2(\tau)\phi_{(1,2)}^2(x_1(\tau))e_1^2(\tau)d\tau \leq \beta_\theta(\hat{\theta}(t) - \hat{\theta}(0)) \leq \beta_\theta\overline{\theta}.$$

Hence,

$$\tilde{V}(t) - \tilde{V}(0) \leq -\chi \int_0^t \tilde{V}(\tau)d\tau + \tilde{\theta}^2 \frac{4c}{a^*}\lambda_{max}^2(P_o)\overline{G}^2 \beta_\theta \overline{\theta} \quad (65)$$

which implies that

$$\begin{aligned} \tilde{V}(t) &\leq \tilde{V}(0) + \tilde{\theta}^2 \frac{4c}{a^*}\lambda_{max}^2(P_o)\overline{G}^2\beta_\theta\overline{\theta} \\ \int_0^t \tilde{V}(\tau)d\tau &\leq \frac{1}{\chi}\tilde{V}(0) + \frac{1}{\chi}\tilde{\theta}^2 \frac{4c}{a^*}\lambda_{max}^2(P_o)\overline{G}^2\beta_\theta\overline{\theta}. \end{aligned} \quad (66)$$

Noting that

$$\min\Big(1, \frac{\lambda_{min}(\tilde{P}_o)}{\lambda_{max}(P_o)}\Big)V \leq \tilde{V} \leq \max\Big(1, \frac{\lambda_{max}(\tilde{P}_o)}{\lambda_{min}(P_o)}\Big)V, \quad (67)$$

the conclusion of Lemma 3 follows. ⋄

Proof of Lemma 4: As proved in Lemma 3, if $\sup_{t\in[0,t_f)} \hat{\theta}(t) \leq \overline{\theta}$, then $V(t)$ and $\int_0^t V(\tau)d\tau$ are bounded on $t \in [0, t_f)$. As in the proof of Lemma 2, this implies the boundedness of all closed-loop signals, thus contradicting the possibility that $t_f < \infty$. Therefore, solutions exist for all time and $t_f = \infty$. Hence, from Lemma 3, $\sup_{t\in[0,\infty)} V(t) < \infty$, and $\int_0^\infty V(t) < \infty$ which, using Barbalat's Lemma, implies that $\lim_{t\to\infty} V(t) = 0$. ⋄

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