Effect of earth’s rotation on angular velocity and acceleration readings of an Inertial Measurement Unit (IMU)

Consider an IMU which is powered on at time $t = 0$ and kept stationary at a point on the earth’s surface. The IMU measures angular velocities and linear accelerations in a right-handed coordinate frame, denoted by $X_{imu}(t)$, fixed to it (the body frame). A stationary coordinate frame with origin at the center of the earth and with positive $z$ axis pointing towards the north pole (and with arbitrarily oriented $x$ and $y$ axes in the equatorial plane of the earth such that the $x$, $y$, and $z$ axes form a right-handed coordinate frame) is denoted $X_{earth,0}$. Consider a coordinate frame $X_{earth}(t)$ that is identical to $X_{earth,0}$ at time $t = 0$ and rotates at a fixed angular velocity $\omega_{earth} \approx 7.2921 \times 10^{-5}$ rad/sec about its $z$ axis. Hence, $X_{earth}(t)$ is related to $X_{earth,0}$ through a rotation matrix

$$R(t) = \begin{bmatrix} \cos(\omega_{earth}t) & \sin(\omega_{earth}t) & 0 \\ -\sin(\omega_{earth}t) & \cos(\omega_{earth}t) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

Since the IMU is stationary at a fixed point on the earth’s surface, the coordinate frame $X_{imu}(t)$ is related to the coordinate frame $X_{earth}(t)$ through a constant rotation matrix $R_0$. Thus, $X_{imu}(t)$ is related to $X_{earth,0}$ through the rotation matrix $R_0R(t)$. Hence, $X_{imu}(t)$ is related to $X_{imu}(0)$ through the rotation matrix $M(t)$ which may be found as$^1$

$$X_{imu}(t) = X_{imu}(0)M(t) \quad (2)$$
$$X_{earth,0}R_0R(t) = X_{earth,0}M(t)R_0 \quad (3)$$
$$\implies M(t) = R_0R(t)R_0^{-1}. \quad (4)$$

The IMU measures angular velocity in its body frame, i.e., it measures a $3 \times 1$ vector $\omega_{imu} = [\omega_{imu,x}, \omega_{imu,y}, \omega_{imu,z}]^T$. The angular velocity measured in the inertial frame $X_{imu}(0)$ is $\omega_{imu,0} = M(t)\omega_{imu}$. Hence$^2$,

$$\dot{M}(t) = S(\omega_{imu,0})M(t) \quad (5)$$
$$= S(M(t)\omega_{imu})M(t) \quad (6)$$
$$= M(t)S(\omega_{imu})M^T(t)M(t) \quad (7)$$
$$= M(t)S(\omega_{imu}) \quad (8)$$

where $S(\omega)$, with $\omega = [\omega_x, \omega_y, \omega_z]^T$ being a $3 \times 1$ vector, is the skew symmetric matrix

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \quad (9)$$

$^1$ $M(t)$ takes vectors from the $X_{imu}(t)$ frame to the $X_{imu}(0)$ frame.

$^2$ We have utilized the property $S(Ra) = RS(a)R^T$ where $R$ is a rotation matrix.
Noting that $\dot{R}(t) = R(t)S(\omega_e)$ where $\omega_e = [0, 0, 1]^T \omega_{earth}$,

$$M(t)S(\omega_{imu}) = \dot{M}(t) = R_0R(t)S(\omega_e)R_0^{-1}$$

$$\implies S(\omega_{imu}) = R_0S(\omega_e)R_0^{-1}. \quad (10)$$

$$\implies S(\omega_{imu}) = R_0S(\omega_e)R_0^{-1}. \quad (11)$$

Thus, $S(\omega_{imu})$ is related to $S(\omega_e)$ through a similarity transformation so that they share the same eigenvalues. Since $S(\omega_e)$ is given by

$$S(\omega_e) = \begin{bmatrix} 0 & -\omega_{earth} & 0 \\ \omega_{earth} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (12)$$

the eigenvalues of $S(\omega_e)$ are $0, j\omega_{earth}$, and $-j\omega_{earth}$. Hence, the angular velocity of the earth can be obtained from the angular velocity readings of the IMU as the imaginary part of the complex conjugate eigenvalues of $S(\omega_{imu})$. Noting that the eigenvalues of $S(\omega)$ given by (9) are $0$ and $\pm j\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$, it follows that the magnitude of $\omega_{imu}$ is equal to $\omega_{earth}$. This is to be expected since the body frame of the IMU being at a fixed orientation with respect to the earth is rotating at the angular speed of the earth around the axis of rotation of the earth. Hence, it can also be shown that $\omega_{imu}$ is oriented along the $z$ axis of the $X_{earth,0}$ coordinate frame (i.e., the rotational axis of the earth). In other words, the direction cosines of the earth’s axis of rotation with respect to the IMU body frame are $\frac{\omega_{imu,x}}{\omega_{earth}}, \frac{\omega_{imu,y}}{\omega_{earth}}$, and $\frac{\omega_{imu,z}}{\omega_{earth}}$. Hence,

$$R_0 \omega_{imu} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_{earth}. \quad (13)$$

However, as expected, (13) does not suffice to uniquely identify $R_0$. This is because angular velocity readings serve to identify only one direction in space (the earth’s rotational axis) and provides no information on orientation in the perpendicular plane to that direction. If the IMU is levelled with respect to the earth’s surface in the sense that the $z$ axis of the IMU body frame passes through the center of the earth, then the latitude $\varphi$ of the point on earth at which the IMU is located is given by 90 degrees minus the angle between the $z$ axis of the $X_{imu}$ coordinate frame and the $z$ axis of the $X_{earth,0}$ coordinate frame, i.e.,

$$\varphi = \frac{\pi}{2} - \cos^{-1}\left(\frac{\omega_{imu,z}}{\omega_{earth}}\right). \quad (14)$$

If, however, the IMU is not levelled with respect to the earth’s surface, the accelerometer readings must be used to find the direction to the earth’s center. With the IMU at rest on the earth’s surface, the accelerometer readings are due to two effects:

1. the acceleration due to gravity directed towards the center of the earth
2. the centripetal acceleration directed towards the earth’s rotational axis and orthogonal to it.
These two acceleration effects are depicted in Figure 1. The magnitude of the acceleration due to gravity is $g \approx 9.8 \text{ m/sec}^2$ and the magnitude of the centripetal acceleration is $\omega_{\text{earth}}^2 r_{\text{earth}} \cos(\varphi)$ where $r_{\text{earth}} \approx 6.378 \times 10^6 \text{ m}$ is the radius of the earth. Hence, the IMU measures an acceleration vector $[a_{\text{imu},x}, a_{\text{imu},y}, a_{\text{imu},z}]^T$ in its body frame with magnitude

$$\sqrt{a_{\text{imu},x}^2 + a_{\text{imu},y}^2 + a_{\text{imu},z}^2} = \sqrt{g^2 + \omega_{\text{earth}}^4 r_{\text{earth}}^2 \cos^2(\varphi) + 2g\omega_{\text{earth}}^2 r_{\text{earth}} \cos^2(\varphi)}$$

(15)

and directed along a vector deflected by an angle $\zeta$ from the direction to the earth’s center where

$$\zeta = \tan^{-1}\left(\frac{\omega_{\text{earth}}^2 r_{\text{earth}} \sin(\varphi) \cos(\varphi)}{g + \omega_{\text{earth}}^2 r_{\text{earth}} \cos^2(\varphi)}\right).$$

(16)

The angle between the net acceleration vector and the net angular velocity vector given by

$$\alpha = \cos^{-1}\left(\frac{\omega_{\text{imu},x} a_{\text{imu},x} + \omega_{\text{imu},y} a_{\text{imu},y} + \omega_{\text{imu},z} a_{\text{imu},z}}{\sqrt{a_{\text{imu},x}^2 + a_{\text{imu},y}^2 + a_{\text{imu},z}^2}}\right).$$

(17)

satisfies

$$\alpha = \frac{\pi}{2} - \varphi + \zeta.$$

(18)
Hence, we obtain the trigonometric equation

\[
\tan(\varphi) = \cot(\alpha - \zeta) = \frac{1 + \tan(\alpha) \frac{\omega_{\text{earth}}^2 r_{\text{earth}} \sin(\varphi) \cos(\varphi)}{g + \omega_{\text{earth}}^2 r_{\text{earth}} \cos^2(\varphi)}}{\tan(\alpha) - \frac{\omega_{\text{earth}}^2 r_{\text{earth}} \sin(\varphi) \cos(\varphi)}{g + \omega_{\text{earth}}^2 r_{\text{earth}} \cos^2(\varphi)}}
\]  

(19)

which, after simplification, yields

\[
\varphi = \tan^{-1} \left( \frac{g + \omega_{\text{earth}}^2 r_{\text{earth}}}{g \tan(\alpha)} \right).
\]  

(20)

Noting that \( g \approx 9.8 \text{ m/sec}^2 \) and \( \omega_{\text{earth}}^2 r_{\text{earth}} \approx 0.034 \text{ m/sec}^2 \), it follows from (20) that \( \zeta \) may be neglected and

\[
\varphi \approx \frac{\pi}{2} - \alpha.
\]  

(21)