**Recipes for Pole Placement**

**Task:** Consider an \( n^{th} \) order linear controllable single-input (i.e., \( u \) being scalar) system \( \dot{x} = Ax + Bu \). With the linear state feedback \( u = Kx \), the closed-loop system is \( \dot{x} = (A + BK)x \). Given a set of \( n \) desired eigenvalues \( p_1, \ldots, p_n \) for the closed-loop system matrix \( A + BK \), find a \( K \) which places the eigenvalues of \( A + BK \) at those desired values.

You can use either of the two recipes given below for this task.

**Recipe 1:**

1. Find the controllability matrix \( \Gamma_n = [B, AB, A^2B, \ldots, A^{n-1}B] \).
2. Find the characteristic polynomial \( p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \) of the matrix \( A \).
3. Find the column vector \( q_1 = (\Gamma_n^T)^{-1}e_n \) where \( e_n = [0, \ldots, 0, 1]^T \) is the \( 1 \times n \) column vector with 1 as the \( n^{th} \) element and zero everywhere else. Define

\[
Q = \begin{bmatrix}
q_1^T \\
q_1^T A \\
q_1^T A^2 \\
\vdots \\
q_1^T A^{n-1}
\end{bmatrix}.
\]

If we perform the coordinate transformation \( \hat{x} = Qx \), then in the new coordinate frame, the system is in controller canonical form.

4. Find the characteristic polynomial corresponding to the desired eigenvalues, i.e., find \( (\lambda + p_1)(\lambda + p_2)\ldots(\lambda + p_n) \). Denote this characteristic polynomial by \( \lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0 \).
5. Let \( \hat{k}_i = a_{i-1} - d_{i-1}, \ i = 1, \ldots, n \). Define \( \hat{K} = [\hat{k}_1, \ldots, \hat{k}_n] \). Then, \( K = \hat{K}Q \).

**Recipe 2:**

1. Find the characteristic polynomial \( p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \) of the matrix \( A \).
2. Define

\[
p_1(\lambda) = \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \ldots + a_2\lambda + a_1 \\
p_2(\lambda) = \lambda^{n-2} + a_{n-1}\lambda^{n-3} + \ldots + a_3\lambda + a_2
\]
3. Define $T = [p_1(A)B, p_2(A)B, \ldots, p_n(A)B]$ and $Q = T^{-1}$. If we perform the coordinate transformation $\hat{x} = Qx$, then in the new coordinate frame, the system is in controller canonical form.

4. Find the characteristic polynomial corresponding to the desired eigenvalues, i.e., find $(\lambda + p_1)(\lambda + p_2) \ldots (\lambda + p_n)$. Denote this characteristic polynomial by $\lambda^n + d_{n-1}\lambda^{n-1} + \ldots + d_1\lambda + d_0$.

5. Let $\hat{k}_i = a_{i-1} - d_{i-1}$, $i = 1, \ldots, n$. Define $\hat{K} = [\hat{k}_1, \ldots, \hat{k}_n]$. Then, $K = \hat{K}Q$.

$$
\begin{align*}
p_3(\lambda) & = \lambda^{n-3} + a_{n-1}\lambda^{n-4} + \ldots + a_4\lambda + a_3 \\
\vdots & \\
p_{n-1}(\lambda) & = \lambda + a_{n-1} \\
p_n(\lambda) & = 1.
\end{align*}
$$