Example of dynamics computation (Euler-Lagrange and Newton-Euler formulations): Revolute-Prismatic (RP) manipulator (Problem 7.8)

Consider the Revolute-Prismatic (RP) manipulator shown in the figure below.

Let the coordinate system of the base frame (frame 0) be such that \( z_0 \) is pointing out of the page and \( x_0 \) is pointing to the right. Then, \( y_0 \) is pointing towards top in the figure. The joint variables are \( q_1 = \theta_1 \) and \( q_2 = d_2 \). Let the masses of the two links be \( m_1 \) and \( m_2 \). Since this is a planar manipulator and rotation is only around the \( z_0 \) axis, only the inertia around the \( z_0 \) axis is relevant; let \( I_{01,z} \) and \( I_{02,z} \) denote the moments of inertia of links 1 and 2, respectively, around the axis pointing out of the page (\( I_{01,z} \) and \( I_{02,z} \) are both defined relative to a coordinate frame with origin at the center of mass of the link).

If the planar motion of the manipulator is in the horizontal plane, then gravity terms are not relevant. If the planar motion of the manipulator is in the vertical plane, then gravity terms need to be considered. Let gravity be in the downward direction in the figure (i.e., in the \(-y_0\) direction). As shown in the figure, the first link has an L-shape; let the longer length of this link be denoted by \( l \) and the shorter length (the offset) by \( a \). Let \( l_{c1} \) denote the distance from the base (origin of frame 0) to the center of mass of link 1. Also, assume that the linkage between links 1 and 2 is such that when joint 2 actuates, it shifts the center of mass of link 2 by distance \( q_2 \).

**Euler-Lagrange formulation to find the dynamics:** The angular velocity Jacobian matrix corresponding to the first link is:

\[
J_{\omega_1} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{bmatrix}
\]  
(1)

Since the second joint is prismatic, the angular velocity Jacobian matrix corresponding to the second link is also:

\[
J_{\omega_2} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{bmatrix}
\]  
(2)

We find the linear velocity Jacobian matrices of the first and second link to be:

\[
J_{v_1} = \begin{bmatrix}
-l_{c1}s_1 & 0 \\
l_{c1}c_1 & 0 \\
0 & 0
\end{bmatrix} ;
J_{v_2} = \begin{bmatrix}
-l_1s_1 - ac_1 - q_2s_1 & c_1 \\
l_1c_1 - as_1 + q_2c_1 & s_1 \\
0 & 0
\end{bmatrix}
\]  
(3)

where \( s_1 = \sin(q_1) \) and \( c_1 = \cos(q_1) \). Hence, the matrix \( D(q) \) is given by:

\[
D(q) = m_1J_{\omega_1}^TJ_{v_1} + m_2J_{\omega_2}^TJ_{v_2} + J_{\omega_1}^TR_1^0I_1(R_1^0)^TJ_{\omega_1} + J_{\omega_2}^TR_2^0I_2(R_2^0)^TJ_{\omega_2}
\]
\[
= \begin{bmatrix}
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{bmatrix}
\]  
(4)
where
\[ d_{11} = m_2a^2 + m_2l_1^2 + 2m_2l_1q_2 + m_1l_1^2 + m_2q_2^2 + I_{1,z} + I_{2,z} \]  
\[ d_{12} = d_{21} = -am_2 \]  
\[ d_{22} = m_2 \]

As described above, since the rotation of both links is only about the \( z_0 \) axis, only the moments of inertia about the axis pointing out of the page are relevant (i.e., \( I_{1,z} \) and \( I_{2,z} \)).

The Christoffel symbols for the manipulator are found as follows:
\[
\begin{align*}
c_{111} &= \frac{1}{2} \frac{\partial d_{11}}{\partial q_1} = 0 \\
c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} = -l_1m_2 - m_2q_2 \\
c_{121} &= c_{211} = \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} = l_1m_2 + m_2q_2 \\
c_{122} &= c_{212} = -\frac{1}{2} \frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2} \frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2} \frac{\partial d_{22}}{\partial q_2} = 0
\end{align*}
\]

The potential energy of the manipulator is given by:
\[ P = m_1gl_1s_1 + m_2g(l_1s_1 + ac_1 + q_2s_1). \]

Hence,
\[ g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \end{bmatrix} = \begin{bmatrix} m_1gl_1c_1 + m_2g(l_1c_1 - as_1 + q_2c_1) \\ m_2gs_1 \end{bmatrix}. \]

We find the \((k, j)^{th}\) elements of the matrix \(C(q, \dot{q})\) to be
\[
\begin{align*}
c_{11} &= c_{211} \dot{q}_2 \\
c_{12} &= c_{121} \dot{q}_1 \\
c_{21} &= c_{112} \dot{q}_1 \\
c_{22} &= 0
\end{align*}
\]

The dynamical equations of the manipulator are given by:
\[ D(q) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + C(q, \dot{q}) \dot{q} + g(q) = \begin{bmatrix} \tau_1 \\ f_2 \end{bmatrix} \]

where \( \tau_1 \) is the applied torque at the first joint (revolute) and \( f_2 \) is the applied force at the second joint (prismatic). Hence, the dynamical equations can be written as:
\[
\begin{align*}
d_{11} \ddot{q}_1 + d_{12} \ddot{q}_2 + c_{211} \dddot{q}_1 + c_{121} \dddot{q}_2 + m_1gl_1c_1 + m_2g(l_1c_1 - as_1 + q_2c_1) &= \tau_1 \\
d_{21} \ddot{q}_1 + d_{22} \ddot{q}_2 + c_{112} \dddot{q}_1 + m_2gs_1 &= f_2.
\end{align*}
\]

**Newton-Euler formulation to find the dynamics:**

- Forward recursion: To assign the Denavit-Hartenberg coordinate frames, pick \( z_0 \) pointing out of the page and \( x_0 \) to the right in the figure (i.e., \( y_0 \) pointing towards top in the figure); pick \( z_1 \) pointing along the actuation axis of the second joint; \( x_1 \) should be picked such that it intersects both \( z_0 \) and \( z_1 \) and is perpendicular to both \( z_0 \) and \( z_1 \). Hence, \( x_1 \) is in the plane of the page. Then, \( y_1 \) can point out of the page. Pick \( z_2 \) along the same direction as \( z_1 \) (with origin of frame 2 at the end-effector) and \( x_2 \) parallel to \( x_1 \). Then, we have \( y_1 \) and \( y_2 \) parallel to \( z_0 \). The Denavit-Hartenberg table for this manipulator is:

<table>
<thead>
<tr>
<th>Link</th>
<th>( a_i )</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a )</td>
<td>90°</td>
<td>0</td>
<td>90° + ( q_1 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>( l_1 + q_2 )</td>
<td>0</td>
</tr>
</tbody>
</table>
Since the rotation is physically around the \( z_0 \) axis, and the axes \( y_1 \) and \( y_2 \) are parallel to \( z_0 \), the angular velocities of both link 1 and link 2 (written relative to the link-fixed frames) are \( \dot{q}_1 \hat{j} \), i.e., \( \omega_1 = \omega_2 = \dot{q}_1 \hat{j} \). Here, the notations \( \hat{i}, \hat{j}, \) and \( \hat{k} \) denote the \( 3 \times 1 \) unit vectors, i.e., \( \hat{i} = [1, 0, 0]^T, \hat{j} = [0, 1, 0]^T, \hat{k} = [0, 0, 1]^T \). Since the base frame is stationary, we have \( a_{c, 0} = a_{e, 0} = 0 \). Looking at the orientation of frame 1, we have \( r_{1, c_1} = l_{c_1} \hat{k} \). Hence, the linear acceleration of the center of mass of link 1 is:

\[
a_{c, 1} = R_0^1 a_{c, 0} + \omega_1 \times r_{1, c_1} + \omega_1 \times (\omega_1 \times r_{1, c_1}) = l_{c_1} \dot{q}_1 \hat{i} - l_{c_1} \dot{q}_1^2 \hat{k}. \tag{16}
\]

Taking the end of link 1 to be the location where it meets the next link (i.e., the tip of the shorter end of the \( L \) shape of link 1; note that the end of the link can be defined to be at any convenient location fixed to the link as long as we keep the equations consistent), we can write \( r_{1, 2} = l_1 \hat{k} + a \hat{i} \). Hence, \( a_{c, 1} \) is

\[
a_{c, 1} = R_0^1 a_{c, 0} + \omega_1 \times r_{1, 2} + \omega_1 \times (\omega_1 \times r_{1, 2}) = l_1 \dot{q}_1 \hat{i} - a \dot{q}_1 \hat{k} - l_1 \dot{q}_1^2 \hat{r} - a \dot{q}_1^2 \hat{r}. \tag{17}
\]

The gravity vector written in frame 1 is given by \( g_1 = -R_0^1 g \hat{j} = -g[c_{1, 0}, s_{1, 0}]^T \). The gravity vector written in frame 2 is also given by \( g_2 = -R_0^2 g \hat{j} = -g[c_{1, 0}, s_{1, 0}]^T \).

The linear acceleration of the center of mass of link 2 is:

\[
a_{c, 2} = R_2^a a_{c, 1} + \omega_2 \times r_{2, c_2} + \omega_2 \times (\omega_2 \times r_{2, c_2}) + R_2^a g = [(R_2^a \omega_1) \times (R_2^a \hat{z}_1)] q_{2, c_2} \hat{j}. \tag{18}
\]

i.e., \( g \) is prismatic or revolute) where \( \hat{z}_1 \) is prismatic or revolute) where \( \hat{z}_1 \) is the \( z_1 \) axis (actuation axis of joint 2) is along the \( z_2 \) direction when seen relative to frame 2. Also, \( r_{2, c_2} = q_2 \hat{k} \).

Hence,

\[
a_{c, 2} = (l_1 + q_2) \dot{q}_1 \hat{i} - a \dot{q}_1 \hat{k} - (l_1 + q_2) \dot{q}_1^2 \hat{k} + a \dot{q}_1^2 \hat{r} + q_2 \hat{k} + q_1 \dot{q}_2 \hat{r}. \tag{19}
\]

- Backward recursion: Start with \( f_3 = \tau_3 = 0 \). Then,

\[
f_2 = m_2a_{c, 2} - g_2 = m_2[(l_1 + q_2) \dot{q}_1 \hat{i} - a \dot{q}_1 \hat{k} - (l_1 + q_2) \dot{q}_1^2 \hat{k} - a \dot{q}_1^2 \hat{r} + q_2 \hat{k} + q_1 \dot{q}_2 \hat{r} + g c_1 \hat{i} + g s_1 \hat{k}]. \tag{20}
\]

\[
\tau_2 = -f_2 \times r_{2, c_2} + I_2 \ddot{\omega}_2 = \{m_2[q_2(l_1 + q_2) \dot{q}_1 - a q_2^2 \dot{q}_1 + q_2 \dot{q}_1 \dot{q}_2 + q_2 g c_1] + I_2 \dot{\omega}_2 \} \hat{j}. \tag{21}
\]

Then,

\[
f_1 = R_2^a f_2 + m_1(a_{c, 1} - g_1) = m_2[(l_1 + q_2) \dot{q}_1 \hat{i} - a \dot{q}_1 \hat{k} - (l_1 + q_2) \dot{q}_1^2 \hat{k} - a \dot{q}_1^2 \hat{r} + q_2 \hat{k} + q_1 \dot{q}_2 \hat{r} + g c_1 \hat{i} + g s_1 \hat{k}] + m_1(l_1 \dot{q}_1 \hat{i} - l_1 \dot{q}_1^2 \hat{k} + g c_1 \hat{i} + g s_1 \hat{k}). \tag{22}
\]

Also, since \( r_{c_2, c_1} = (l_{c_1} - l_1) \hat{k} - a \hat{i} \),

\[
\tau_1 = R_2^a \tau_2 = f_1 \times r_{1, c_1} + (R_2^a f_2) \times r_{2, c_2} + I_1 \ddot{\omega}_1 + \omega_1 \times (I_1 \ddot{\omega}_1) = \{m_2[q_2(l_1 + q_2) \dot{q}_1 - a q_2^2 \dot{q}_1 + q_2 \dot{q}_1 \dot{q}_2 + q_2 g c_1] + I_2 \dot{\omega}_2 \hat{k} + l_1 \dot{f}_{1, x} - (l_{c_1} - l_1) f_{2, x} - a f_{2, z} + I_1 \dot{\omega}_1 \} \hat{j}. \tag{23}
\]

where

\[
\dot{f}_{1, x} = m_2[(l_1 + q_2) \dot{q}_1 - a \dot{q}_1^2 + \dot{q}_1 \dot{q}_2 + g c_1] + m_1(l_{c_1} \dot{q}_1 + g c_1) \tag{24}
\]

\[
f_{2, z} = m_2[a \dot{q}_1 - (l_1 + q_2) \dot{q}_1^2 + \dot{q}_1 + g s_1] \tag{25}
\]

Since the actuation of the first joint is along \( z_0 = q_1 \), the actuated joint torque for the first joint (revolute) is \( \tau_{1, y} \). Since the actuation of the second joint is along \( z_1 = q_2 \), the actuated joint force for the second joint is \( f_{2, z} \). Another way to see which joint forces are the externally actuated forces is to look at \( \dot{f}^T_{i, z} \hat{z}_{i-1} \) or \( \tau^T_{i, z} \hat{z}_{i-1} \) (depending on whether joint is prismatic or revolute) where \( \hat{z}_{i-1} \) denotes the axis \( z_{i-1} \) written relative to frame \( i \); here, \( \tau^T_{1, z} \hat{z}_{0} \) and \( \dot{f}^T_{i, z} \hat{z}_{2} \) is \( f_{2, z} \).

Therefore, the dynamics equations are:

\[
m_2[q_2(l_1 + q_2) \dot{q}_1 - a q_2 \dot{q}_1^2 + q_2 \dot{q}_1 \dot{q}_2 + q_2 g c_1] + l_{c_1} \dot{f}_{1, x} - (l_{c_1} - l_1) f_{2, x} - a f_{2, z} + (I_1 \dot{z} + I_2 \dot{z}_2) \dot{q}_1 = u_1 \tag{26}
\]

\[
m_2[-a \dot{q}_1 - (l_1 + q_2) \dot{q}_1^2 + \dot{q}_1 + g s_1] = u_2 \tag{27}
\]

where \( u_1 \) is the actuated torque for the first joint and \( u_2 \) is the actuated force for the second joint, respectively.

Substituting (23)-(25) into (26), we get the dynamics equation (14). Also, we see that (27) is the same as the dynamics equation (15).