1. Show that the following sets with the + and * operations specified below are vector spaces:

(a) set of all \(2 \times 3\) matrices with real entries over the field \(\mathbb{R}\) with the usual matrix addition and scalar multiplication.

(b) \(p_2 = \) the set of all real polynomials of degree two or less over \(\mathbb{R}\) with the usual polynomial addition and scalar multiplication

(c) \(C[a, b] = \) the set of real-valued continuous functions on \([a, b]\) over \(\mathbb{R}\) with addition \(h(x) = f(x) + g(x), x \in [a, b]\) and scalar multiplication \(q[x] = cf(x), x \in [a, b]\).

2. Let \(C^2[a, b] = \) the set of all real-valued functions \(f(x)\) defined on \([a, b]\) where \(f(x), f'(x),\) and \(f''(x)\) are continuous on \([a, b]\). This set with the usual addition of functions and scalar multiplication is a vector space. Which of the following two subsets of \(C^2[-1, 1]\) are vector spaces?

(a) \(S = \{f(x) \in C^2[-1, 1] \mid f''(x) + f(x) = 0, -1 \leq x \leq 1\}\)

(b) \(S = \{f(x) \in C^2[-1, 1] \mid f''(x) + f(x) = x^2, -1 \leq x \leq 1\}\)

3. Show that the following functions are inner products:

(a) \(V = \mathbb{R}^n.\) Let \(<x, y> = x^Ty.\)

(b) \(V = p_2.\) Let \(<p, q> = p(0)q(0) + p(1)q(1) + p(2)q(2).\)

(c) \(V = p_2.\) Let \(<p, q> = \int_0^1 p(x)q(x)dx.\)

4. The following system is a controllable and observable system:

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x.
\end{align*}
\]

Design an observer with poles at \(-10 \pm j10\). Also, design a controller to place the poles of the system at \(-3 \pm j3\). Draw a block diagram of your designed system.

5. Show that the pair \(\left\{ \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} , \begin{bmatrix} b \\ 0 \end{bmatrix} \right\}\) is controllable if and only if the following statement is true:

\(\{A, b\}\) is controllable and \(\begin{bmatrix} A & b \\ C & 0 \end{bmatrix}\) has full rank (Hint: Use the PBH test).