EL7253 Homework 1 Solutions

1. To verify that a set $V$ with some defined addition and multiplication operations is a vector space over a field of scalars $F$, we need to show the following properties:

   (a) $x + y \in V$ for any elements $x$ and $y$ in $V$; $ax \in V$ for any $x \in V$ and any scalar $a \in F$
   (b) $x + y = y + x$ for any elements $x$ and $y$ in $V$
   (c) $x + (y + z) = (x + y) + z$ for any elements $x$, $y$, and $z$ in $V$
   (d) A zero element $\theta$ exists in $V$ such that $\theta + x = x$ for any $x \in V$
   (e) For any $x \in V$, an element $-x$ exists in $V$ such that $x + (-x) = \theta$
   (f) $a(bx) = (ab)x$ for any elements $a$ and $b$ in $F$ and any element $x \in V$
   (g) $a(x + y) = ax + ay$ for any $a \in F$ and any $x, y \in V$
   (h) $(a + b)x = ax + bx$ for any $a, b \in F$ and any $x \in V$
   (i) An element 1 exists in $F$ such that $1x = x$ for any $x \in V$.

These properties can be demonstrated easily for the following sets:

   (a) set of all $(2 \times 3)$ matrices with real entries over the field $\mathbb{R}$ with the usual matrix addition and scalar multiplication: Here, the zero element is the zero matrix of dimension $2 \times 3$. The 1 element is simply the real number 1.
   (b) $p_2$ = the set of all real polynomials of degree two or less over $\mathbb{R}$ with the usual polynomial addition and scalar multiplication: Here, the zero element is the polynomial 0 (note that this is simply a polynomial with all coefficients being zero). The 1 element is the real number 1.
   (c) $C[a, b]$ = the set of real-valued continuous functions on $[a, b]$ over $\mathbb{R}$ with addition $h(x) = f(x) + g(x)$, $x \in [a, b]$ and scalar multiplication $q(x) = cf(x)$, $x \in [a, b]$. Here, the zero element is the function that is zero on the entire interval $[a, b]$. The 1 element is the real number 1.

2. Let $C^2[a, b]$ = the set of all real-valued functions $f(x)$ defined on $[a, b]$ where $f(x)$, $f'(x)$, and $f''(x)$ are continuous on $[a, b]$. This set with the usual addition of functions and scalar multiplication is a vector space. We need to determine which of the following two subsets of $C^2[-1, 1]$ are vector spaces:

   (a) $S_1 = \{ f(x) \in C^2[-1, 1] \mid f''(x) + f(x) = 0, -1 \leq x \leq 1 \}$
   (b) $S_2 = \{ f(x) \in C^2[-1, 1] \mid f''(x) + f(x) = x^2, -1 \leq x \leq 1 \}$

It can be verified based on the properties of vector spaces listed in Problem 1 above that the subset in (a) is a vector space since it satisfies all the properties required of a vector space. The subset in (b) is however not a vector space since it does not satisfy the property that given any two elements $f_1$ and $f_2$ in the set, the sum $f_1 + f_2$ should also be in the set. In this case, if $f_1$ and $f_2$ are in the set $S_1$, then we have:

\[
\begin{align*}
f''_1(x) + f_1(x) &= x^2 \\
f''_2(x) + f_2(x) &= x^2.
\end{align*}
\]

Hence, defining $f_s = f_1 + f_2$, we have $f''_s(x) + f_s(x) = 2x^2$ which is not equal to $x^2$. Hence, $f_s$ is not in the set $S_2$. Therefore, $S_2$ is not a vector space.

3. Given a vector space $V$ over the field of reals, a function $\langle x, y \rangle$ from $V \times V$ to $\mathbb{R}$ is an inner product if it satisfies the following properties:

   (a) $\langle x, x \rangle$ is greater than or equal to zero for all $x \in V$ and furthermore $\langle x, x \rangle$ is equal to zero if and only if $x = \theta$, i.e., the zero element in $V$. 

These properties can be verified for the following functions:

(a) \( V = \mathbb{R}^n \) with \( < x, y > = x^T y \). Here, \( < x, x > \) is equal to \( x^T x \) which is non-negative and is zero only when \( x \) is the zero element in \( V \). Also, since \( x^T y = y^T x \), we get \( < x, y > = < y, x > \).

Furthermore, \((a x, y ) = a < x, y >\) and \( x^T (y + z) = x^T y + x^T z\).

(b) \( V = p_2 \) with \( < p, q > = p(0)q(0) + p(1)q(1) + p(2)q(2) \). Here, \( < p, p > = [p(0)]^2 + [p(1)]^2 + [p(2)]^2 \) which is non-negative and is zero only when \( p \) is zero, i.e., the zero element in \( V \). The properties (b), (c), and (d) above also follow directly from the given definition of the function \( <.,.> \).

4. With the observer gains being \( G = [g_1, g_2]^T \), the matrix \( A + GC \) becomes \( \begin{bmatrix} g_1 & 1 \\ 9 + g_2 & 0 \end{bmatrix} \). The values of \( g_1 \) and \( g_2 \) to place the eigenvalues of \( A + GC \) at \(-10 \pm j10\) are \(-20\) and \(-209\), respectively. Hence, the desired observer is given by:

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 - 20(\dot{x}_1 - x_1) \\
\dot{x}_2 &= 9\dot{x}_1 - u - 209(\dot{x}_1 - x_1)
\end{align*}
\]

(1)

The controller gain vector \( K = [k_1, k_2] \) to place the eigenvalues of \( A + BK \) at \(-3 \pm j3\) is \([27,6]\). The block diagram of the overall system is shown below.

5. We can utilize the PBH rank test. By the PBH rank test, the controllability of the pair \( \left( \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} \right) \) is equivalent to the full rank condition on the matrix \( \begin{bmatrix} sI - A & 0 \\ C & sI \end{bmatrix} \) for all \( s \) in the complex plane;
we know that it is sufficient to check the rank condition for all $s$ that are eigenvalues of the matrix
\[
\begin{bmatrix}
A & 0 \\
C & 0
\end{bmatrix},
\]
i.e., the value 0 and the eigenvalues of $A$. At $s = 0$, the rank of
\[
\begin{bmatrix}
sI - A & 0 & b \\
C & sI & 0
\end{bmatrix}
\]
is equal to the rank of
\[
\begin{bmatrix}
A & b \\
C & 0
\end{bmatrix}.
\]
At a non-zero $s$ equal to an eigenvalue of $A$, the rank of
\[
\begin{bmatrix}
sI - A & 0 & b \\
C & sI & 0
\end{bmatrix}
\]
is equal to $p + \text{rank}(sI - A : b)$ where $p$ is the number of rows in $C$, i.e., the dimension of $C$ is $p \times n$.

Thus, by looking at the cases of $s$ equal to 0 and $s$ equal to an eigenvalue of $A$, it follows that the pair \{\[
\begin{bmatrix}
A & 0 \\
C & 0
\end{bmatrix}, \begin{bmatrix}
b \\
0
\end{bmatrix}
\]\} is controllable if and only if the pair \{\!
\begin{bmatrix}
A & b \\
C & 0
\end{bmatrix}\!
\} is controllable and the matrix
\[
\begin{bmatrix}
A & b \\
C & 0
\end{bmatrix}
\]
has full rank.

An alternative approach to prove the given statement is to compute the controllability matrix for the pair \{\[
\begin{bmatrix}
A & 0 \\
C & 0
\end{bmatrix}, \begin{bmatrix}
b \\
0
\end{bmatrix}
\]\} as
\[
C = \begin{bmatrix}
b & Ab & A^2b & \ldots & A^{n+p-1}b \\
0 &Cb & CAb & \ldots & CA^{n+p-2}b
\end{bmatrix}
\]
which is equal to
\[
C = \begin{bmatrix}
A & b \\
C & 0
\end{bmatrix} \begin{bmatrix}
0 & b & Ab & \ldots & A^{n+p-2}b \\
I & 0 & 0 & \ldots & 0
\end{bmatrix}
\]
The given statement follows directly from (3) since the matrix $C$ has full rank if and only if the two matrices $\begin{bmatrix}
A & b \\
C & 0
\end{bmatrix}$ and $\begin{bmatrix}
0 & b & Ab & \ldots & A^{n+p-2}b \\
I & 0 & 0 & \ldots & 0
\end{bmatrix}$ have full rank, i.e., the matrix $\begin{bmatrix}
A & b \\
C & 0
\end{bmatrix}$ has full rank and the pair \{\!
\begin{bmatrix}
A & b \\
C & 0
\end{bmatrix}\!
\} is controllable.