1. We are given the linear time-invariant system

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

with

\[
A = \begin{bmatrix}
-1 & 3 & 1 & 0 \\
-1 & 1 & 1 & 0 \\
0 & 2 & 0 & 0 \\
-4 & 4 & 4 & -8 \\
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
0 \\
0 \\
2 \\
1 \\
\end{bmatrix}
\]
\[
C = [1, 1, -1, 2].
\]

Using the Popov-Belevitch-Hautus (PBH) test, we need to check the controllability and observability of this system and find the uncontrollable modes (eigenvalues), if any, and the unobservable modes, if any, of the system.

**Controllability Test:** We need to check the rank of the matrix \([sI - A : B]\) at the eigenvalues of \(A\). In the given system, the eigenvalues of \(A\) are \(-8, 0, 0, 0\). The rank of the matrix \([sI - A : B]\) is 3 at \(s = -8\) and 4 at \(s = 0\). Hence, the system has an uncontrollable mode at \(s = -8\).

**Observability Test:** We need to check the rank of the matrix \([sI - A : C]\) at the eigenvalues of \(A\), i.e., at \(-8, 0, 0, 0\). The rank of the matrix \([sI - A : C]\) is 4 at \(s = -8\) and 3 at \(s = 0\). Hence, the system has an unobservable mode at \(s = 0\).

2. The element in the \(B\) matrix corresponding to the last row of every Jordan block for each eigenvalue should be non-zero in order to obtain a controllable system. A Jordan block is a submatrix (that appears as a diagonal block in \(A\)) of \(A\) and has the same value on all its diagonal elements and has 1’s on the upper diagonal. The value that is repeated on the diagonal elements of the Jordan block is an eigenvalue of the \(A\) matrix. The given \(A\) matrix has 5 Jordan blocks. Furthermore, if an eigenvalue has more than one Jordan block, then the system is controllable only if the number of inputs is at least the same number as the number of Jordan blocks for that eigenvalue. In the given \(A\) matrix, the eigenvalue 1 has two Jordan blocks (one block of dimension 3 \(\times\) 3 and another of dimension 1 \(\times\) 1), the eigenvalue 2 has one Jordan block (of dimension 2 \(\times\) 2), and the eigenvalue 3 has two Jordan blocks (both of dimension 1 \(\times\) 1). Hence, if \(B\) is a single column, the system is not controllable. In particular, consider the seventh and eighth state variables \(x_7\) and \(x_8\); the dynamics of these state variables are of the form \(\dot{x}_7 = 3x_7 + b_7u\) and \(\dot{x}_8 = 3x_8 + b_8u\). Define \(z_7 = b_8x_7 - b_7x_8\). Then, \(\dot{z}_7 = 3z_7\). Note that a state transformation can be performed such that \(z_7\) is one of the new state coordinates. In these transformed coordinates, \(z_7\) cannot be affected by the control input. Hence, it is seen that the given system is not controllable. However, if \(B\) has multiple columns, i.e., the system has multiple inputs, then, as described above, the system can be made controllable with an appropriate choice of the elements of \(B\).

3. We wish to prove that complete controllability is equivalent to controllability to zero. It is obvious from the definition of controllability that complete controllability indeed implies controllability to zero as a special case. To prove the converse, consider the following initial state: \(\hat{x}_0 = x_0 - \Phi(t_0, t_f)x_f\). If
the system is controllable to zero, then a \( \hat{u} \) exists such that \( \dot{x}_0 \) can be transferred to zero in finite time. Therefore,

\[
0 = \Phi(t_f, t_0)\hat{x}_0 + \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)\hat{u}(\tau)\,d\tau
\]

\[
= \Phi(t_f, t_0)[x_0 - \Phi(t_0, t_f)x_f] + \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)\hat{u}(\tau)\,d\tau
\]

\[
= \Phi(t_f, t_0)x_0 - x_f + \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)\hat{u}(\tau)\,d\tau. \quad (1)
\]

Hence,

\[
x_f = \Phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)\hat{u}(\tau)\,d\tau. \quad (2)
\]

Therefore, there exists an input signal, namely \( \hat{u} \), such that the state \( x_0 \) is transferred to the state \( x_f \) in a finite time. Note that we have relied on the invertibility property of \( \Phi(t_0, t_f) \) in this proof. In particular, if the system is time-invariant, then the invertibility of \( \Phi(t_0, t_f) \) follows directly from the nonsingularity of \( e^{At} \) for all \( t \). In general, it can be shown that the property that \( \Phi(t_0, t_f) \) is nonsingular holds for all linear continuous-time systems.

The equivalence of complete controllability and controllability to zero is not true in the discrete-time case since the transition matrix may not be invertible. For instance, consider the system

Any initial state, e.g., \( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \) can be transferred to the origin in one time step using the control law \( u(k) = (-\alpha - \beta)\delta(k) \). However, starting from the origin, it is not possible to reach an arbitrary state since the value of \( x_2(k) \) would be 0 for all \( k \geq 0 \) regardless of the applied input signal. Hence, for discrete-time systems, controllability to the origin does not imply complete controllability.

4. We are given

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]

\[
C = [0, 1, 0, -1].
\]

The controllable subspace is found to be the subspace spanned by the vectors

\[
g_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad g_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}; \quad g_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (4)
\]

Defining \( V_1 = [g_1, g_2, g_3] \), it is verified that the controllable subspace is \( A \)-invariant by noting that the columns of the matrix \( AV_1 \) lie in the controllable subspace found above. Equivalently, it is seen that the rank of \([V_1, AV_1]\) is the same as the rank of \( V_1 \) implying that the subspace spanned by the columns of \( AV_1 \) is a subset of \( V_1 \).

The rank of the matrix with rows \( C, CA, CA^2 \), and \( CA^3 \) is found to be 1. Hence, the unobservable subspace has dimension 3. The unobservable subspace is found to be the subspace spanned by the vectors

\[
h_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad h_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad h_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad (5)
\]
Defining $V_2 = [h_1, h_2, h_3]$, it is seen that the rank of $[V_2, AV_2]$ is the same as the rank of $V_2$; hence, the subspace spanned by the columns of $AV_2$ is a subset of $V_2$, i.e., the columns of the matrix $AV_2$ lie in the unobservable subspace found above. Hence, it is verified that the unobservable subspace is $A$-invariant.

Finding the intersection of the controllable subspace and the unobservable subspace, it is found that the controllable subspace and the unobservable subspace are identical for this system. Hence, to perform Kalman decomposition, we can utilize the vectors $h_1$, $h_2$, $h_3$, and one additional vector to complete $\mathbb{R}^4$. In this case, we can pick the fourth vector to be $[0, -1, 0, 1]^T$. Hence, define

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \quad (6)$$

Hence, after a similarity transformation, we get

$$\hat{A} = Z^{-1}AZ = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 3 & -1 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{B} = Z^{-1}B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{C} = CZ = [0, 0, 0, -2]. \quad (7)$$

The first three state coordinates represent the controllable and unobservable part and the last state coordinate represents the uncontrollable and observable part.

5. A Matlab program to perform Kalman decomposition for the system is given at: [http://crrl.poly.edu/7253/kalman_decomposition.html](http://crrl.poly.edu/7253/kalman_decomposition.html)