1. We have \( B = [B_1, B_2] \) with \( B_1 = [0, 0, 1, 1, 1]^T \) and \( B_2 = [0, 1, 0, 0, 1]^T \). Evaluating \( B_1, B_2, AB_1, AB_2, A^2B_1, \ldots \), we note that the rank of \([B_1, B_2, AB_1, AB_2, A^2B_1] \) is 5. Hence, the controllability indices are \( \mu_1 = 3 \) and \( \mu_2 = 2 \). Defining

\[
\bar{C} = [B_1, AB_1, A^2B_1, B_2, AB_2] = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 4 & 6 & 1 & 1 \\
1 & 1 & 3 & 0 & 0 \\
1 & 1 & 6 & 0 & 1 \\
1 & 4 & 10 & 1 & 1 \\
\end{bmatrix}, \tag{1}
\]

we get

\[
\bar{C}^{-1} = \begin{bmatrix}
-4 & -1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 \\
-7 & -3 & -3 & -1 & 4 \\
-3 & 0 & -1 & 1 & 0 \\
\end{bmatrix}. \tag{2}
\]

Since the controllability indices for this system are 3 and 2, we need the third and fifth rows of \( \bar{C}^{-1} \). Hence, define

\[
q_1 = [1, 0, 0, 0, 0] \tag{3}
\]

\[
q_2 = [-3, 0, -1, 1, 0]. \tag{4}
\]

The similarity transformation matrix to transform the system into the multi-input version of the controllable canonical form is defined as

\[
P = \begin{bmatrix}
q_1 \\
q_1A \\
q_1A^2 \\
q_2 \\
q_2A \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
-3 & 0 & -1 & 1 & 0 \\
-7 & -3 & -4 & 0 & 4 \\
\end{bmatrix}. \tag{5}
\]

After the similarity transformation, we get

\[
A_c = PAP^{-1} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
2 & 3 & 0 & 0 & 1 \\
\end{bmatrix} \tag{6}
\]

\[
B_c = PB = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}. \tag{7}
\]

which is in controllable canonical form, \( \dot{z} = A_cz + B_cu \) where \( z = Px \). The matrices \( A_c \) and \( B_c \) can be
written as \( A_c = \overline{A}_c + \overline{B}_c A_m \) and \( B_c = \overline{B}_c B_m \) where

\[
\overline{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\] (8)

\[
\overline{B}_c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\] (9)

\[
A_m = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{bmatrix}
\] (10)

\[
B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\] (11)

Note that \( \overline{A}_c \) and \( \overline{B}_c \) depend only on the controllability indices of the system and the matrices \( A_m \) and \( B_m \) are obtained from \( A_c \) and \( B_c \), respectively, by taking the \( \mu_1^{th} \) and \( (\mu_1 + \mu_2)^{th} \) rows, i.e., the third and fifth rows. Utilizing a control gain matrix \( K_c \), we can assign the third and fifth rows of the closed-loop system matrix \( A_c + B_c K_c \), i.e., if we define \( A_{des} = \overline{A}_c + \overline{B}_c A_{des,m} \), then we can make the closed-loop system matrix \( A_c + B_c K_c \) to be equal to \( A_{des} \) by picking \( K_c = B_m^{-1}(A_{des,m} - A_m) \). Note that \( A_{des} \) has the same structure of controllability indices as \( A_c \), i.e., \( A_{des} \) differs from \( A_c \) only in the \( \mu_1^{th} \) and \( (\mu_1 + \mu_2)^{th} \) rows, i.e., the third and fifth rows.

The desired pole locations of the closed-loop system are given as \(-2, -2 \pm j, -3 \pm 2j\). We can pick \( A_{des} \) with the structure \( \overline{A}_c + \overline{B}_c A_{des,m} \) as defined above in multiple ways to achieve the desired pole locations. In particular (it can be seen that there are other possible ways as well to assign the desired eigenvalues), two possible choices of \( A_{des} \) are:

- \( A_{des} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -130 & -229 & -166 & -62 & -16 \end{bmatrix} \)
- \( A_{des} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -26 & -25 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -5 & -4 \end{bmatrix} \)

In the first choice of \( A_{des} \) shown above, we have

\[
A_{des,m} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -130 & -229 & -166 & -62 & -16 \end{bmatrix}
\] (12)

\[
K_c = \begin{bmatrix} 0 & -2 & -1 & 0 & 0 \\ -132 & -232 & -166 & -62 & -13 \end{bmatrix}
\] (13)

\[
K_x = K_c P = \begin{bmatrix} -4 & -2 & -3 & 0 & 2 \\ -319 & -193 & -284 & -62 & 180 \end{bmatrix}
\] (14)

where \( K_c \) and \( K_x \) are the control gain matrices written in the transformed state representation \( z \) and in the original state representation \( x \), respectively, i.e., the control law is \( u = K_c z = K_x x \).
In the second choice of $A_{des}$ that was shown above, we have

$$A_{des,m} = \begin{bmatrix} -26 & -25 & -8 & 0 & 0 \\ 0 & 0 & 0 & -5 & -4 \end{bmatrix}$$

(15)

$$K_c = \begin{bmatrix} -26 & -27 & -9 & -1 & 0 \\ -2 & -3 & 0 & -5 & -5 \end{bmatrix}$$

(16)

$$K_x = K_cP = \begin{bmatrix} -77 & -27 & -35 & -1 & 27 \\ 42 & 12 & 22 & -5 & -17 \end{bmatrix}$$

(17)

where $K_c$ and $K_x$ are the control gain matrices written in the transformed state representation $z$ and in the original state representation $x$, respectively, i.e., the control law is $u = K_cz = K_xx$.

2. As in the solution of Problem 1 above, we first put the given multi-input system into the multi-input version of the controllable canonical form. We have $B = [B_1, B_2]$ with $B_1 = [1, 0, 0, 0, 0]^T$ and $B_2 = [0, 0, 1, 0, 0]^T$. Evaluating $B_1, B_2, AB_1, AB_2, A^2B_1, \ldots$, we find that the rank of $[B_1, B_2, AB_1, AB_2]$ is 4. $A^2B_1$ is in the span of $\{B_1, B_2, AB_1, AB_2\}$, and the rank of $[B_1, B_2, AB_1, AB_2, A^2B_2]$ is 5. Hence, the controllability indices are $\mu_1 = 2$ and $\mu_2 = 3$. Defining

$$\mathcal{C} = [B_1, AB_1, B_2, AB_2, A^2B_2] = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 & 11 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(18)

we get

$$\mathcal{C}^{-1} = \begin{bmatrix} 1 & -4 & 0 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(19)

Since the controllability indices for this system are 2 and 3, we need the second and fifth rows of $\mathcal{C}^{-1}$. Hence, define

$$q_1 = [0, 1, 0, -1, 0]$$

(20)

$$q_2 = [0, 0, 0, 0, 1].$$

(21)

The similarity transformation matrix to transform the system into the multi-input version of the controllable canonical form is defined as

$$P = \begin{bmatrix} q_1 \\ q_1A \\ q_2 \\ q_2A \\ q_2A^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(22)

After the similarity transformation, we get

$$A_c = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -4 & -4 \end{bmatrix}$$

(23)

$$B_c = PB = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(24)
which is in controllable canonical form, $\dot{z} = A_c z + B_c u$ where $z = P x$. The matrices $A_c$ and $B_c$ can be written as $A_c = \bar{A}_c + \bar{B}_c A_m$ and $B_c = \bar{B}_c B_m$ where

$$\bar{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (25)

$$\bar{B}_c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (26)

$$A_m = \begin{bmatrix} 1 & 4 & 2 & -1 & 0 \\ 0 & 2 & -4 & -4 \end{bmatrix}$$  \hspace{1cm} (27)

$$B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$  \hspace{1cm} (28)

Note that, as in the solution of Problem 1 above, $A_c$ and $B_c$ depend only on the controllability indices of the system and the matrices $A_m$ and $B_m$ are obtained from $A_c$ and $B_c$, respectively, by taking the $\mu_1^{th}$ and $(\mu_1 + \mu_2)^{th}$ rows, i.e., the second and fifth rows. Utilizing a control gain matrix $K_c$, we can assign the second and fifth rows of the closed-loop system matrix $A_c + B_c K_c$, i.e., if we define $A_{des} = \bar{A}_c + \bar{B}_c A_{des,m}$, then we can make the closed-loop system matrix $A_c + B_c K_c$ to be equal to $A_{des}$ by picking $K_c = B_m^{-1} (A_{des,m} - A_m)$. Note that $A_{des}$ has the same structure of controllability indices as $A_c$, i.e., $A_{des}$ differs from $A_c$ only in the $\mu_1^{th}$ and $(\mu_1 + \mu_2)^{th}$ rows, i.e., the second and fifth rows. The desired pole locations of the closed-loop system are given as $-1, -2, -2, -2 \pm j$. There are many ways to pick $A_{des}$ with the structure $\bar{A}_c + \bar{B}_c A_{des,m}$ to achieve these desired pole locations. In particular, two possible choices of $A_{des}$ are:

- $A_{des} =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -20 & -56 & -61 & -33 & -9 \end{bmatrix}$$  \hspace{1cm} (29)

- $A_{des} =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -5 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -8 & -5 \end{bmatrix}$$

In the first choice of $A_{des}$ shown above, we have

$$A_{des,m} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -20 & -56 & -61 & -33 & -9 \end{bmatrix}$$  \hspace{1cm} (29)

$$K_c = \begin{bmatrix} -1 & -4 & -1 & 1 & 0 \\ -20 & -56 & -63 & -29 & -5 \end{bmatrix}$$  \hspace{1cm} (30)

$$K_x = K_c P = \begin{bmatrix} -4 & -1 & 0 & 2 & 3 \\ -56 & -20 & -5 & -9 & -12 \end{bmatrix}$$  \hspace{1cm} (31)

where $K_c$ and $K_x$ are the control gain matrices written in the transformed state representation $z$ and in the original state representation $x$, respectively, i.e., the control law is $u = K_c z = K_x x$. 


In the second choice of $A_{des}$ that was shown above, we have

$$A_{des,m} = \begin{bmatrix} -5 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & -8 & -5 \end{bmatrix}$$ (32)

$$K_c = \begin{bmatrix} -6 & -8 & -2 & 1 & 0 \\ 0 & 0 & -6 & -4 & -1 \end{bmatrix}$$ (33)

$$K_x = K_c P = \begin{bmatrix} -8 & -6 & 0 & 7 & 6 \\ 0 & 0 & -1 & -4 & -7 \end{bmatrix}$$ (34)

where $K_c$ and $K_x$ are the control gain matrices written in the transformed state representation $z$ and in the original state representation $x$, respectively, i.e., the control law is $u = K_c z = K_x x$.

3. (a) To find the similarity transformation matrix to put the given system into the observable canonical form, we can consider the dual system with $\tilde{A} = A^T$ and $\tilde{B} = C^T$. For the given $A$ and $C$ matrices, we have

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$ (35)

$$\tilde{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$ (36)

Defining $\tilde{B}_1$ and $\tilde{B}_2$ to be the first and second columns, respectively, of $\tilde{B}$, we find that the rank of $[\tilde{B}_1, \tilde{B}_2, A\tilde{B}_1, A\tilde{B}_2]$ is 4. Hence, the controllability indices are $\mu_1 = 2$ and $\mu_2 = 2$. Hence, define

$$\mathcal{C} = [\tilde{B}_1, \tilde{A}\tilde{B}_1, \tilde{B}_2, A\tilde{B}_2] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$ (37)

$$\mathcal{C}^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0.5 & -0.5 & 0.5 \\ 1 & -1.5 & 0.5 & 0.5 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$ (38)

Since the controllability indices were found above to be 2 and 2, we need the second and fourth rows of $\mathcal{C}^{-1}$. Hence, define

$$q_1 = [0, 0.5, -0.5, 0.5]$$ (39)

$$q_2 = [0, -1, 1, 0].$$ (40)

The similarity transformation matrix to transform the pair $\{\tilde{A}, \tilde{B}\}$ into the multi-input version of the controllable canonical form is defined as

$$P = \begin{bmatrix} q_1 \\ q_1\tilde{A} \\ q_2 \\ q_2\tilde{A} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & -0.5 & 0.5 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}.$$ (41)

The similarity transformation matrix to put the original system into observable canonical form is given by $P_o = ZP^{-T}$ where

$$Z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$ (42)

5
Hence,

$$P_o = ZP^{-T} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & -2 & -2 & 2
\end{bmatrix}. \tag{43}$$

After performing a similarity transformation of the given system by $P_o$, we get

$$A_o = P_oA P_o^{-1} = \begin{bmatrix}
-2 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 \\
-3 & 0 & 4 & 1 \\
-2 & 0 & 0 & 0
\end{bmatrix}$$

$$B_o = P_oB = \begin{bmatrix}
0 \\
-1 \\
2 \\
-4
\end{bmatrix}$$

$$C_o = CP_o^{-1} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{44}$$

which is in the observable canonical form.

To find $G$ to place the eigenvalues of $A - GC$, we perform a similarity transformation of the dual pair $\{A, B\}$ using the matrix $P$ found above. After the similarity transformation, we get

$$\hat{A}_c = P\hat{A}P^{-1} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 4 & 0 & 2 \\
0 & 0 & 0 & 1 \\
-2 & -3 & 0 & -2
\end{bmatrix} \tag{45}$$

$$\hat{B}_c = P\hat{B} = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix} \tag{46}$$

which is in controllable canonical form. The matrices $A_c$ and $B_c$ can be written as $\hat{A}_c = \overline{A}_c + \overline{B}_cA_m$ and $\hat{B}_c = \overline{B}_cB_m$ where

$$\overline{A}_c = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{47}$$

$$\overline{B}_c = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix} \tag{48}$$

$$A_m = \begin{bmatrix}
0 & 4 & 0 & 2 \\
-2 & -3 & 0 & -2
\end{bmatrix} \tag{49}$$

$$B_m = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \tag{50}$$

Utilizing a control gain matrix $\hat{K}_c$, we can assign the second and fourth rows of the closed-loop system matrix $\hat{A}_c + \hat{B}_c\hat{K}_c$, i.e., if we define $\hat{A}_{des} = \overline{A}_c + \overline{B}_cA_{des,m}$, then we can make the closed-loop system matrix $\hat{A}_c + \hat{B}_c\hat{K}_c$ to be equal to $\hat{A}_{des}$ by picking $\hat{K}_c = B_m^{-1}(A_{des,m} - A_m)$. Note that $\hat{A}_{des}$ has the same structure of controllability indices as $A_c$, i.e., $A_{des}$ differs from $A_c$ only in the $\mu_1^{th}$ and $(\mu_1 + \mu_2)^{th}$
rows, i.e., the second and fourth rows. The desired pole locations of the closed-loop system are given as $-1, -2, -2 \pm j$. There are many ways to pick $\tilde{A}_{des}$ with the structure $\tilde{A}_c + \tilde{B}_c A_{des,m}$ to achieve these desired pole locations. In particular, two possible choices of $\tilde{A}_{des}$ are:

- $\tilde{A}_{des} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -10 & -23 & -19 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -5 & -4 \end{bmatrix}$
- $\hat{A}_{des} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In the first choice of $\tilde{A}_{des}$ shown above, we have

$$A_{des,m} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -10 & -23 & -19 & -7 \end{bmatrix}, \quad \tilde{K}_c = \begin{bmatrix} 0 & -4 & 1 & -2 \\ -8 & -20 & -19 & -5 \end{bmatrix}, \quad \hat{K} = \tilde{K}_c P = \begin{bmatrix} -2 & -3 & 1 & -2 \\ -5 & 0 & -15 & -19 \end{bmatrix}. \quad (51)$$

Hence,

$$G = -\hat{K}^T = \begin{bmatrix} 2 & 5 \\ 3 & 0 \\ -1 & 15 \\ 2 & 19 \end{bmatrix}. \quad (54)$$

In the second choice of $\tilde{A}_{des}$ that was shown above, we have

$$A_{des,m} = \begin{bmatrix} -2 & -3 & 0 & 0 \\ 0 & 0 & -5 & -4 \\ -2 & -7 & 0 & -2 \\ 2 & 3 & -5 & -2 \end{bmatrix}, \quad \tilde{K}_c = \begin{bmatrix} -2 & -6 & 1 & -6 \\ -2 & 11 & -6 & 11 \end{bmatrix}, \quad \hat{K} = \tilde{K}_c P = \begin{bmatrix} -2 & -6 & 1 & -6 \\ -2 & 11 & -6 & 6 \end{bmatrix}. \quad (57)$$

Hence,

$$G = -\hat{K}^T = \begin{bmatrix} 2 & 2 \\ 6 & -11 \\ -1 & 6 \\ 6 & -6 \end{bmatrix}. \quad (58)$$

(b) To find the reduced-order observer, the first step is to perform a coordinate transformation so that in the new coordinates, the output $y$ corresponds to the first state coordinate. This can be accomplished using a similarity transformation with a matrix $T$ of the form $T = \begin{bmatrix} C \\ M \end{bmatrix}$ where $M$ is a $2 \times 4$ matrix such that $T$ is of full rank. In this case, we can pick $M$ to be, for instance,

$$M = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (59)$$

Performing the similarity transformation with matrix $T$, we get

$$w = Tx = \begin{bmatrix} Cx \\ Mx \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (60)$$
with the dynamics of $w$ being of the form

$$
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} \\
\bar{A}_{21} & \bar{A}_{22}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} +
\begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} u
$$

(61)

where

$$
\bar{A}_{11} =
\begin{bmatrix}
2 & -1 \\
1.5 & -1.5
\end{bmatrix}
$$

(62)

$$
\bar{A}_{12} =
\begin{bmatrix}
0 & 2 \\
-0.5 & 1
\end{bmatrix}
$$

(63)

$$
\bar{A}_{21} =
\begin{bmatrix}
0.5 & -1.5 \\
0.5 & -0.5
\end{bmatrix}
$$

(64)

$$
\bar{A}_{22} =
\begin{bmatrix}
0.5 & 1 \\
0.5 & 1
\end{bmatrix}
$$

(65)

$$
\bar{B}_1 =
\begin{bmatrix}
2
\end{bmatrix}
$$

(66)

$$
\bar{B}_2 =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

(67)

The reduced-order observer design is now given as

$$
\dot{z} = Fz + \bar{G}_1 u + \bar{G}_2 y
$$

$$
\hat{w}_2 = Ly + z
$$

(68)

where $F$ is a stable matrix and the following equations hold:

$$
F = \bar{A}_{22} - L\bar{A}_{12}
$$

$$
\bar{G}_1 = \bar{B}_2 - L\bar{B}_1
$$

$$
\bar{G}_2 = \bar{A}_{21} - L\bar{A}_{11} + FL.
$$

(69)

It is specified that it is desired that all the poles of the observer error dynamics should be placed at $-1$. Hence, $L$ should be picked such that both the eigenvalues of $\bar{A}_{22} - L\bar{A}_{12}$ are at $-1$. Picking, for example, $L =
\begin{bmatrix}
2 \\
1.5
\end{bmatrix}$, we get $\bar{A}_{22} - L\bar{A}_{12} = -I$. Hence, this choice of $L$ can be utilized to place the observer error dynamics poles at the desired locations. The matrices $F$, $\bar{G}_1$, and $\bar{G}_2$ are found to be:

$$
F =
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
$$

(70)

$$
\bar{G}_1 =
\begin{bmatrix}
-4 \\
-3
\end{bmatrix}
$$

(71)

$$
\bar{G}_2 =
\begin{bmatrix}
-1 & -1 \\
-2.5 & 0.5
\end{bmatrix}
$$

(72)

Thereafter, the estimate of the state $x$ can be obtained by using $\hat{x} = P_{xy} + Q\hat{w}_2$ where $T^{-1} = [P \mid Q]$. With the choice of $T$ as defined above, the matrices $P$ and $Q$ become:

$$
P =
\begin{bmatrix}
0 & 1 \\
0.5 & -0.5 \\
0.5 & -0.5 \\
0 & 0
\end{bmatrix}
$$

(73)

$$
Q =
\begin{bmatrix}
0 & 0 \\
0.5 & 0 \\
-0.5 & 0 \\
0 & 1
\end{bmatrix}
$$

(74)