1. (a) We are given the linear system:
\[
\begin{align*}
\dot{x}_1 &= x_2 + u + d \\
\dot{x}_2 &= u
\end{align*}
\]
where \(d\) is a constant disturbance signal. The given system corresponds to
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
(1)
The exosystem formulation for the given disturbance signal is:
\[
x_0 = d, \quad A_0 = A_d = 0, \quad E = [1, 0]^T
\]
(2)
Since there is no non-zero reference signal provided for the output \(y\) to track (i.e., the performance objective corresponds to state regulation to zero), we do not need a reference model \(\dot{x}_m = A_m x_m\).
The metasystem state is given by \(\bar{x} = \begin{bmatrix} x \\ x_0 \end{bmatrix}\). Here, the metasystem is a third-order system. The dynamics of the metasystem can be written as:
\[
\dot{\bar{x}} = \bar{A} \bar{x} + \bar{B} u
\]
(3)
where
\[
\bar{A} = \begin{bmatrix} A & E \\ 0 & A_0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}.
\]
(4)
(b) The output is given to be \(y = x_1 - x_2\) and is required to track a given \(y_{\text{ref}}(t) = 1\). The model for the reference signal is \(\dot{x}_m = 0\) since the reference signal \((y_{\text{ref}} = x_m = 1)\) is a constant. To account for the reference signal, we need to redefine the state coordinates to include the tracking error. Here, \(C = [1, -1]\) and we can pick a transformation matrix
\[
T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]
(5)
The first row in \(T\) follows from the output definition. The definition of the second row is arbitrary as long as it is not linearly dependent on the first row. Picking the second row of \(T\) to be orthogonal to \(C\) usually simplifies the numerical calculations.
Define \(\tilde{x} = Tx + T_m x_m\) with \(T_m = [-1, 0]^T\), i.e., we have defined
\[
\tilde{x} = \begin{bmatrix} x_1 - x_2 - y_{\text{ref}} \\ x_1 + x_2 \end{bmatrix}
\]
(6)
The dynamics of \(\tilde{x}\) are of the form \(\dot{\tilde{x}} = \tilde{A} \tilde{x} + \tilde{B} u + \tilde{E} x_0\) where
\[
\begin{align*}
x_0 &= \begin{bmatrix} d \\ x_m \end{bmatrix} \\
\tilde{A} &= \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \\
\tilde{B} &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
\tilde{E} &= \begin{bmatrix} 1 & -0.5 \\ 1 & -0.5 \end{bmatrix}.
\end{align*}
\]
(7)
Here, since both the disturbance signal and the reference signal are constants, $A_0$ is simply a $2 \times 2$ matrix with all zero elements. The metasystem state is given by $\mathbf{x} = \begin{bmatrix} \tilde{x} \\ x_0 \end{bmatrix}$. Here, the metasystem is a fourth-order system. The dynamics of the metasystem can be written as

$$\dot{\mathbf{x}} = \mathbf{A}_0 \mathbf{x} + B u$$

where

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} & \mathbf{E} \\ 0 & \mathbf{A}_0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix}.$$

(c) The output is given to be $y = x_1 - x_2$ and is required to track a given $y_{ref}(t) = \sin(t)$. The model for this reference signal is $\dot{x}_m = A_m x_m$, $y_{ref} = C_m x_m$ with

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$C_m = [1, 0]$$

Hence, $A_0$ is given by

$$A_0 = \begin{bmatrix} A_d & 0 \\ 0 & A_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

To account for the reference signal, we need to redefine the state coordinates to include the tracking error. Here, $C = [1, -1]$ and we can pick a transformation matrix

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$ 

The first row in $T$ follows from the output definition. The definition of the second row is arbitrary as long as it is not linearly dependent on the first row. Picking the second row of $T$ to be orthogonal to $C$ usually simplifies the numerical calculations.

Define $\hat{x} = Tx + T_m x_m$ with

$$T_m = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

The dynamics of $\hat{x}$ are of the form

$$\dot{\hat{x}} = \mathbf{A} \hat{x} + B \tilde{u} + \tilde{E} x_0$$

where

$$x_0 = \begin{bmatrix} d \\ x_m \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\tilde{E} = \begin{bmatrix} 1 & -0.5 & -1 \\ 1 & -0.5 & 0 \end{bmatrix}.$$ 

The metasystem state is given by $\mathbf{\tilde{x}} = \begin{bmatrix} \tilde{x} \\ x_0 \end{bmatrix}$. Here, the metasystem is a fifth-order system. The dynamics of the metasystem can be written as

$$\dot{\mathbf{\tilde{x}}} = \mathbf{A} \mathbf{\tilde{x}} + B u$$
where

\[ \overline{A} = \begin{bmatrix} \hat{A} & \hat{E} \\ 0 & A_0 \end{bmatrix}, \]

\[ \overline{B} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}. \]  

(16)

2. The dynamics of the given system can be written as \( \dot{x} = Ax + Bu + Edx_d \) where \( x = [p - p_{ref}, v - v_{ref}]^T \), \( x_d = v_{ref} \), and

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad E_d = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}. \]  

(17)

The gain matrix \( K \) to place the eigenvalues of \( A + BK \) at the given locations \(-1, -2\) is given by \( K = [-2, -2.5] \). With \( x_d = v_{ref} \) being a constant, the choice of \( G_0 \) to make \( y = p - p_{ref} = Cx = [1, 0]x \) go to zero at steady state is given by

\[ G_0 = -B^#E_d \]  

(18)

where

\[ B^# = [C(A + BK)^{-1}B]^{-1}C(A + BK)^{-1} \]  

(19)

Hence,

\[ B^# = [3, 1] \]

\[ G_0 = 0.5. \]  

(20)

Applying the control law \( u = Kx + G_0x_d \) makes the output \( y \) go to zero asymptotically as \( t \to \infty \) inspite of the (constant) disturbance input \( x_d = v_{ref} \). Note that in this case, the control law \( u = Kx + G_0x_d \) actually exactly cancels out the constant disturbance \( x_d \) since \( BG_0 + E_d = 0 \).