1. Consider a linear system given by:
\[
\begin{align*}
\dot{x}_1 &= x_2 + u \\
\dot{x}_2 &= u
\end{align*}
\]

Design the optimal linear control law to minimize the performance objective given by
\[
V = \int_t^\infty [x_1^2(\tau) + x_2^2(\tau) + u^2(\tau) + x_1(\tau)u(\tau)]d\tau.
\]
Note that this performance objective includes a cross-term between the state and the input.

2. Consider the system
\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u
\]
with \(x = [x_1, x_2, x_3]^T\). Consider the performance cost
\(J = \int_t^\infty (x_1^2 + x_2^2 + ru^2)\) with \(r > 0\) being a parameter. Using the “are” function in MATLAB, find the optimal state-feedback control that minimizes this cost functional for \(r = 1\) and \(r = 10\). What happens to the size of the control gain as the parameter \(r\) is made very small? How about when \(r\) is made very large?

Note: The “are” function in MATLAB can be used to solve the algebraic Riccati equation (ARE). Type “help are” at the MATLAB command prompt to read about this function.

3. Consider a system of form \(\dot{x} = Ax + Bu + Fv, y = Cx + w\) where
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1, 0].
\]
The noise \(w\) and \(v\) are specified to be uncorrelated white noise with spectral densities (covariance matrices) \(W\) and \(V\), respectively.

(a) Write the optimal observer (Kalman filter) for this system if \(\alpha = 1\), \(\beta = 3\), \(V = 1\), and \(W = 1\).

(b) Show that if \(V\) and \(W\) are both scaled by the same constant, (i.e., replaced by \(\tilde{V} = \rho V\) and \(\tilde{W} = \rho W\) with \(\rho\) being a positive constant), then the observer gain of the Kalman filter does not change. Hence, the optimal observer gain vector only depends on the ratio \(\frac{V}{W}\). Plot the elements of the optimal observer gain vector as a function of \(\frac{V}{W}\).