1. Consider the following second order differential equation:

\[ \epsilon \ddot{x}(t) + (2 + t)\dot{x}(t) + x(t) = 0 \; ; \; x(0) = c_1 \text{ and } \dot{x}(0) = c_2. \]

Find the solution of the above differential equation and show that

\[ \lim_{t \to 0} (\lim_{\epsilon \to 0} \dot{x}(t)) \neq \lim_{\epsilon \to 0} (\lim_{t \to 0} \dot{x}(t)). \]

2. Consider

Write the state differential equations for this system using the capacitor voltages as states. Furthermore, assume \( C_2 \ll C_1 \). Show that this system of differential equations can be put in the standard singularly perturbed model. What does the slow subsystem correspond to (physically)?

3. Consider

\[ \dot{x} = x z^3 \]
\[ \epsilon \dot{z} = -z - x^\frac{4}{3} + \frac{4}{3} \epsilon x^\frac{16}{3}. \]

Obtain the reduced order models of this system using two techniques: (1) two-time scale analysis; (2) Integral manifold approach.

4. Consider

\[
\begin{bmatrix}
\dot{x} \\
\epsilon \dot{z}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[ J = \frac{1}{2} \int_0^\infty (q_1 x^2 + q_2 z^2 + ru^2) dt \]

where \( q_1, q_2, r > 0 \).

(a) Find the optimal solution.
(b) Find the near-optimal solution (i.e., decomposition into fast and slow time scales).
(c) Compare (a) and (b).

5. Try to find the roots of \( \epsilon z^5 + z - 1 = 0 \) in terms of an asymptotic expansion in \( \epsilon \).