Overview of Kalman Filter

Consider a linear system of form

\[
\begin{align*}
\dot{x} &= Ax + Bu + Fv \\
y &= Cx + w
\end{align*}
\]

(1)

with \(v\) and \(w\) being Gaussian zero-mean white noise random signals. A full order observer for this system is given by

\[
\dot{\hat{x}} = A\hat{x} + Bu + G(y - C\hat{x}).
\]

(2)

\(v\) is known as the process noise and \(w\) is known as the measurement noise. Defining

\[
\begin{align*}
E\{v(t)v^T(\tau)\} &= V(t)\delta(t - \tau) \\
E\{w(t)w^T(\tau)\} &= W(t)\delta(t - \tau) \\
E\{v(t)w^T(\tau)\} &= X(t)\delta(t - \tau),
\end{align*}
\]

(3)

the optimal observer gain is given by \(G = PC^T W^{-1} + FXW^{-1}\) where \(P\) is the positive definite matrix solution of the Riccati equation

\[
\tilde{A}P + PA^T - PC^T W^{-1}CP + F\tilde{V} F^T = 0
\]

(4)

where

\[
\begin{align*}
\tilde{A} &= A - FXW^{-1}C \\
\tilde{V} &= V - XW^{-1}X^T.
\end{align*}
\]

(5)

If the cross spectral density \(X(t)\) is zero (i.e., if the process noise and the measurement noise are uncorrelated), then \(\tilde{A} = A\) and \(\tilde{V} = V\) and the Riccati equation for \(P\) is simply

\[
AP + PA^T - PC^T W^{-1}CP + FVF^T = 0
\]

(6)

and the optimal observer gain is given by \(G = PC^T W^{-1}\).