Kalman Decomposition

Given a linear time-invariant system of the form

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx,
\end{align*}
\]

(1)

with \(x \in \mathbb{R}^n\), \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), and \(C \in \mathbb{R}^{p \times n}\), the controllable subspace and the unobservable subspace are found as follows:

- **Controllable subspace** \(\mathcal{C}\): the span of the columns of \([B, AB, A^2B, \ldots, A^{n-1}B]\)

- **Unobservable subspace** \(\mathcal{O}\): the set of column vectors \(q\) that are such that

\[
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix} q = 0,
\]

(2)

i.e., such that \(Cq = 0\), \(CAq = 0\), \ldots, \(CA^{n-1}q = 0\).

The controllable subspace \(\mathcal{C}\) and the unobservable subspace \(\mathcal{O}\) can both be shown to be \(A\)-invariant as follows:

- If \(q\) is a vector in the controllable subspace \(\mathcal{C}\), then it is, by definition, in the span of the columns of \([B, AB, A^2B, \ldots, A^{n-1}B]\). Then, \(Aq\) should be in the span of the columns of \([AB, A^2B, A^3B, \ldots, A^nB]\). Since \(A^n\) can be written as a linear combination of \(I, A, \ldots, A^{n-1}\) by Cayley-Hamilton theorem, we see that \(Aq\) is in the span of the columns of \([B, AB, A^2B, \ldots, A^{n-1}B]\), i.e., \(Aq\) is also in the controllable subspace. Hence, the controllable subspace \(\mathcal{C}\) is \(A\)-invariant.

- If \(q\) is a vector in the unobservable subspace \(\mathcal{O}\), then, by definition, \(Cq = 0\), \(CAq = 0\), \ldots, \(CA^{n-1}q = 0\). Consider the vector \(\tilde{q} = Aq\). Then, \(C\tilde{q} = CAq = 0\), \(CA\tilde{q} = CA^2q = 0\), \ldots, \(CA^{n-2}\tilde{q} = CA^{n-1}q = 0\). Also, \(CA^{n-1}\tilde{q} = CA^nq\). By Cayley-Hamilton theorem, \(A^n\) can be written as a linear combination of \(I, A, \ldots, A^{n-1}\). Hence, \(CA^nq\) can be written as a linear combination of \(Cq, CAq, \ldots, CA^{n-1}q\); therefore, \(CA^nq = 0\). Hence, \(CA^{n-1}\tilde{q} = 0\). Therefore, \(Aq\) is also in the unobservable subspace. Hence, the unobservable subspace is \(A\)-invariant.

Also, since the intersection of any two \(A\)-invariant subspaces is also \(A\)-invariant, note that the subspace \(\mathcal{C} \cap \mathcal{O}\) formed as the intersection of the controllable subspace \(\mathcal{C}\) and the unobservable subspace \(\mathcal{O}\) is also \(A\)-invariant.

The procedure for Kalman decomposition is as follows:

- Find a linearly independent set of vectors that span \(\mathcal{C} \cap \mathcal{O}\), i.e., the intersection of the controllable and unobservable subspaces. Form a matrix \(Z_1\) with its columns being this linearly independent set of vectors.

- Find a linearly independent set of vectors that along with the columns of \(Z_1\) would span the controllable subspace \(\mathcal{C}\). Form a matrix \(Z_2\) with its columns being this linearly independent set of vectors.

- Find a linearly independent set of vectors that along with the columns of \(Z_1\) would span the unobservable subspace \(\mathcal{O}\). Form a matrix \(Z_3\) with its columns being this linearly independent set of vectors.

- Find a linearly independent set of vectors that along with the columns of \([Z_1, Z_2, Z_3]\) would span \(\mathbb{R}^n\). Form a matrix \(Z_4\) with its columns being this linearly independent set of vectors.

- Define \(Z = [Z_1; Z_2; Z_3; Z_4]\).

- Do a change of coordinates (similarity transformation) of the given system using the matrix \(Z\), i.e., define \(\bar{A} = Z^{-1}AZ\), \(\bar{B} = Z^{-1}B\), and \(\bar{C} = CZ\). In the new coordinates, the system is in Kalman decomposed form.
Denote the numbers of columns in $Z_1$, $Z_2$, $Z_3$, and $Z_4$ by $n_1$, $n_2$, $n_3$, and $n_4$, respectively. We have $n_i \geq 0$, $i = 1, \ldots, 4$, and $\sum_{i=1}^{4} n_i = n$. Denote the matrix $Z^{-1}$ in the form

$$Z^{-1} = \begin{bmatrix} \hat{Z}_1 \\ \hat{Z}_2 \\ \hat{Z}_3 \\ \hat{Z}_4 \end{bmatrix}$$

(3)

where the numbers of rows of $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Z}_3$, and $\hat{Z}_4$ are $n_1$, $n_2$, $n_3$, and $n_4$, respectively.

Since $Z^{-1}Z$ is the identity matrix, we have the equations:

$$\hat{Z}_i Z_i = I_{n_i \times n_i} \quad \text{for } i = 1, \ldots, 4$$

(4)

$$\hat{Z}_i Z_j = 0_{n_i \times n_j} \quad \text{for } i = 1, \ldots, 4 \text{ and } j = 1, \ldots, 4 \text{ such that } i \neq j$$

(5)

where $I_{n_i \times n_i}$ denotes the identity matrix of dimension $n_i \times n_i$ and $0_{n_i \times n_j}$ denotes the zero matrix of dimension $n_i \times n_j$.

From the definitions of $\overline{A}$, $\overline{B}$, and $\overline{C}$, we have

$$\overline{A} = Z^{-1} AZ = \begin{bmatrix} \hat{Z}_1 AZ_1 & \hat{Z}_1 AZ_2 & \hat{Z}_1 AZ_3 & \hat{Z}_1 AZ_4 \\ \hat{Z}_2 AZ_1 & \hat{Z}_2 AZ_2 & \hat{Z}_2 AZ_3 & \hat{Z}_2 AZ_4 \\ \hat{Z}_3 AZ_1 & \hat{Z}_3 AZ_2 & \hat{Z}_3 AZ_3 & \hat{Z}_3 AZ_4 \\ \hat{Z}_4 AZ_1 & \hat{Z}_4 AZ_2 & \hat{Z}_4 AZ_3 & \hat{Z}_4 AZ_4 \end{bmatrix}$$

(6)

$$\overline{B} = Z^{-1} B = \begin{bmatrix} \hat{Z}_1 B \\ \hat{Z}_2 B \\ \hat{Z}_3 B \\ \hat{Z}_4 B \end{bmatrix}$$

(7)

$$\overline{C} = CZ = \begin{bmatrix} CZ_1 & CZ_2 & CZ_3 & CZ_4 \end{bmatrix}.$$ 

(8)

The controllable subspace is given by the span of the columns of $[Z_1, Z_2]$. Since the columns of $B$ are definitely in the controllable subspace, the columns of $Z$ are in the span of the columns of $[Z_1, Z_2]$. Also, by (5), the matrices $\hat{Z}_2 Z_1$, $\hat{Z}_3 Z_2$, $\hat{Z}_4 Z_1$, and $\hat{Z}_4 Z_2$ are all zero. Hence, we see that $\hat{Z}_3 B$ and $\hat{Z}_4 B$ are zero matrices. Hence, from (7), $\overline{B}$ is of the form

$$\overline{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$

(9)

where $B_1 = \hat{Z}_1 B$ and $B_2 = \hat{Z}_2 B$.

Since the columns of $C^T$ are orthogonal to any vector in the unobservable subspace, and since the unobservable subspace is given by the span of the columns in $[Z_1, Z_3]$, we also see that $CZ_1$ and $CZ_3$ are zero matrices, i.e., from (8), $\overline{C}$ is of the form

$$\overline{C} = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix}.$$ 

(10)

where $C_2 = CZ_2$ and $C_4 = CZ_4$.

The columns of the matrix $Z_1$ span the $A$-invariant subspace $C \cap \overline{B}$. Hence, the columns of $AZ_1$ are in the span of the columns of $Z_1$. Since, by (5), the matrices $\hat{Z}_2 Z_1$, $\hat{Z}_3 Z_1$, and $\hat{Z}_4 Z_1$ are all zero, we get

$$\hat{Z}_2 AZ_1 = 0 ; \quad \hat{Z}_3 AZ_1 = 0 ; \quad \hat{Z}_4 AZ_1 = 0.$$ 

(11)

The 0’s in the third and fourth submatrices of the right hand side of (9) denote zero matrices of appropriate dimensions, i.e., $0_{n_3 \times m}$ and $0_{n_4 \times m}$. The 0’s appearing in the following equations also similarly denote zero matrices of the appropriate dimensions.
The controllable subspace is given by the span of the columns of \([Z_1, Z_2]\). Since the controllable subspace is \(A\)-invariant, the columns of \(AZ_2\) are also in the controllable subspace, i.e., in the span of the columns of \([Z_1, Z_2]\). Since, by (5), the matrices \(\hat{Z}_3Z_1, \hat{Z}_4Z_2, \hat{Z}_4Z_1, \) and \(\hat{Z}_4Z_2\) are all zero, we get

\[ \hat{Z}_3AZ_2 = 0 ; \hat{Z}_4AZ_2 = 0. \]  

(12)

The unobservable subspace is given by the span of the columns of \([Z_1, Z_3]\). Since the unobservable subspace is \(A\)-invariant, the columns of \(AZ_3\) are also in the unobservable subspace, i.e., in the span of the columns of \([Z_1, Z_3]\). Since, by (5), the matrices \(\hat{Z}_3Z_1, \hat{Z}_2Z_3, \hat{Z}_4Z_1, \) and \(\hat{Z}_4Z_3\) are all zero, we get

\[ \hat{Z}_2AZ_3 = 0 ; \hat{Z}_4AZ_3 = 0. \]  

(13)

Hence, using (6), (11), (12), and (13), \(A\) is of the form

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
0 & A_{22} & 0 & A_{24} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix}
\]  

(14)

where \(A_{11} = \hat{Z}_1AZ_1, A_{12} = \hat{Z}_1AZ_2, A_{13} = \hat{Z}_1AZ_3, A_{14} = \hat{Z}_1AZ_4, A_{22} = \hat{Z}_2AZ_2, A_{24} = \hat{Z}_2AZ_4, A_{33} = \hat{Z}_3AZ_3, A_{34} = \hat{Z}_3AZ_4, \) and \(A_{44} = \hat{Z}_4AZ_4\).

Hence, denoting the transformed coordinates by

\[
\begin{bmatrix}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4
\end{bmatrix}
\]  

(15)

where \(\pi_1 \in \mathbb{R}^{n_1}, \pi_2 \in \mathbb{R}^{n_2}, \pi_3 \in \mathbb{R}^{n_3}, \) and \(\pi_4 \in \mathbb{R}^{n_4}, \) the system dynamics in the transformed coordinates are of the form

\[
\begin{align*}
\dot{\pi}_1 &= A_{11}\pi_1 + A_{12}\pi_2 + A_{13}\pi_3 + A_{14}\pi_4 + B_1u \\
\dot{\pi}_2 &= A_{22}\pi_2 + A_{24}\pi_4 + B_2u \\
\dot{\pi}_3 &= A_{33}\pi_3 + A_{34}\pi_4 \\
\dot{\pi}_4 &= A_{44}\pi_4 \\
y &= C_2\pi_2 + C_4\pi_4.
\end{align*}
\]  

(16) \quad (17) \quad (18) \quad (19) \quad (20)

Hence, \(\pi_1\) denotes the controllable and unobservable part of the system, \(\pi_2\) denotes the controllable and observable part of the system, \(\pi_3\) denotes the uncontrollable and unobservable part of the system, and \(\pi_4\) denotes the uncontrollable and observable part of the system.