Observer Designs: Full-Order and Reduced-Order

Given a linear time-invariant system of form
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\] (1)
with \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), and \(C \in \mathbb{R}^{q \times n}\), the observer design problem is to construct a dynamic observer such that we can estimate the entire state \(x\) based on measurement of the output \(y\).

There are two basic approaches: full-order observer and reduced-order observer.

**Full-order Observer:** The observer is constructed as
\[
\dot{\hat{x}} = A\hat{x} + Bu - G(C\hat{x} - y).
\] (2)

With the observer error defined as \(e = \hat{x} - x\), we get \(\dot{e} = (A - GC)e\).

The dual of the controller pole placement problem is the observer pole placement problem. To determine the gain matrix \(G\) so as to place the poles at a set of desired pole locations, we use the duality principle to convert the observer pole placement problem into a controller pole placement problem, then use any of the controller pole placement algorithms to determine the gain matrix in controller context, and then finally transform back the gain matrix into the observer context.

**Reduced-order Observer:** The basic idea of reduced-order observer is that since \(y\) is directly measured, there must be a part of the state that is effectively directly measured; hence, we should be able to build an observer of lower dimension than the full-order observer.

If a certain subset of the state is directly measured such as, for instance, \(C = [I_q \mid 0_{q \times (n-q)}]\), then \(x\) has two parts \(x_1 \in \mathbb{R}^q\) and \(x_2 \in \mathbb{R}^{n-q}\) with \(x_1\) being directly measured. The dynamics of the system can be decomposed as
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u
\] (3)

In this case, \(y = x_1\) and we can build a reduced-order observer as
\[
\begin{align*}
\dot{z} &= Fz + \overline{G}_1 u + \overline{G}_2 y \\
\dot{x}_2 &= Ly + z
\end{align*}
\] (4)

where \(F\) is a stable matrix and the following equations hold:
\[
\begin{align*}
F &= A_{22} - L A_{12} \\
\overline{G}_1 &= B_2 - L B_1 \\
\overline{G}_2 &= A_{21} - L A_{11} + FL
\end{align*}
\] (5)

Note that the first equation above is essentially a pole placement problem (i.e., find \(L^T\) to place poles of \(A_{22}^T - A_{12}^T L^T\)); once \(L\) is found using the first equation, \(\overline{G}_1\) and \(\overline{G}_2\) are found using the second and third equations, respectively. It can be shown (using the PBH test for example) that the observability of the pair \((A_{22}, A_{12})\) follows from observability of the pair \((A, C)\) for the given \(C\).
matrix above. If instead of \( y = x_1 \), we have \( y = C_1 x_1 \) with \( C_1 \) being an invertible matrix, then the above reduced-order observer is still valid with the associated equations generalized as

\[
F = A_{22} - LC_1 A_{12} \\
\mathcal{G}_1 = B_2 - LC_1 B_1 \\
\mathcal{G}_2 C_1 = A_{21} - LC_1 A_{11} + FL C_1.
\] (6)

Note that the choice of \( L \) is a pole placement problem. We can choose \( L \) to place the poles of \( F = A_{22} - LA_{12} \) wherever we want if the pair \( (A_{22}, C_1 A_{12}) \) is observable. It can be shown that observability of the pair \( (A_{22}, C_1 A_{12}) \) is guaranteed if the pair \( (A, C) \) is observable, i.e., if the system is observable.

If instead of \( y \) depending on a specific subset of the state, we have, in general, \( y = C x \), then we can still define a reduced-order observer as follows. First, note that \( C \) can be considered as having full row rank \( q \) (i.e., the separate outputs are linearly independent – if this does not hold, then we can simply consider a linearly independent subset of the outputs). We define a transformation matrix of the form

\[
T = \begin{bmatrix} C \\ M \end{bmatrix}
\] (7)

where \( M \) is of dimension \( (n - q) \times n \) and is such that \( T \) is of full rank. Defining \( w = T x \) which is of form

\[
w = T x = \begin{bmatrix} C x \\ M x \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\] (8)

the dynamics of \( w \) are of the form

\[
\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u
\] (9)

and the previously described reduced-order observer design can be applied in the coordinates \( w \) since \( y = w_1 \). Thereafter, the estimate of the state \( x \) can be obtained by using \( \hat{x} = Py + Q \hat{w}_2 \) where \( T^{-1} = [P \mid Q] \).