Transfer functions (from $y$ to $u$) for dynamic output feedback controllers (full-order or reduced-order observer + controller)

Consider a system of form
\[
\dot{x} = Ax + Bu \\
y = Cx.
\] (1)

Consider a state-feedback controller $u = Kx$ that stabilizes this system, i.e., the eigenvalues of $A + BK$ are all in the open left half plane. If an observer (either full order or reduced order) is designed for this system so as to construct an estimate $\hat{x}$ for $x$, then the output-feedback controller formed by combining this observer with the given state-feedback controller is given by $u = K\hat{x}$.

**Transfer function from $y$ to $u$ for output feedback controller implemented using full-order observer:**

A full-order observer is designed as
\[
\dot{\hat{x}} = A\hat{x} + Bu + G(y - C\hat{x}).
\] (2)

Then, since $u = K\hat{x}$, we have
\[
\dot{\hat{x}} = A\hat{x} + BK\hat{x} + G(y - C\hat{x}).
\] (3)

Therefore, taking the Laplace transform, we get
\[
s\hat{X}(s) = A\hat{X}(s) + BK\hat{X}(s) + G(Y(s) - C\hat{X}(s)).
\] (4)

Hence,
\[
\hat{X}(s) = (sI - A - BK + GC)^{-1}GY(s).
\] (5)

Since $u = \hat{x}$, we get
\[
U(s) = K(sI - A - BK + GC)^{-1}GY(s).
\] (6)

Hence, the transfer function from $y$ to $u$ is $H_u(s) = K(sI - A - BK + GC)^{-1}G$.

Alternatively, the transfer function $H_u(s)$ from $y$ to $u$ can be found by writing the observer-controller combination in a state space representation. The combination of the full-order observer and the controller is given by
\[
\dot{\hat{x}} = A\hat{x} + BK\hat{x} + G(y - C\hat{x})
\] (7)
\[
u = K\hat{x}.
\] (8)

This system can be written in state space form as
\[
\dot{\hat{x}} = A_u\hat{x} + B_u y
\] (9)
\[
u = C_u\hat{x} + D_u y
\] (10)

where $A_u = A + BK - GC$, $B_u = G$, $C_u = K$, and $D_u = 0$. Note that the state of this dynamic system is $\hat{x}$, the input is $y$, and the output is $u$. The transfer function from $y$ to $u$ is therefore $H_u(s) = C_u(sI - A_u)^{-1}B_u + D_u$.

**Transfer function from $y$ to $u$ for output feedback controller implemented using reduced-order observer:** Defining $T = \begin{bmatrix} C \\ M \end{bmatrix}$ with $M$ being such that $T$ is a full-rank matrix, and denoting
$T^{-1}$ to be of the form $[P \ Q]$, a reduced-order observer for the system is given by $\dot{x} = T^{-1}\dot{w} = [P \ Q]\dot{w}$ with

$$\dot{w} = \begin{bmatrix} y \\ \dot{w}_2 \end{bmatrix}$$

(11)

where

$$\dot{w}_2 = Ly + z$$

$$\dot{z} = Fz + \overline{G}_1u + \overline{G}_2y.$$  

(12)

Hence,

$$\dot{W}_2(s) = LY(s) + Z(s)$$

$$sZ(s) = FZ(s) + \overline{G}_1U(s) + \overline{G}_2Y(s).$$

(13)

Therefore,

$$Z(s) = (sI - F)^{-1}[\overline{G}_1U(s) + \overline{G}_2Y(s)]$$

$$\dot{W}_2(s) = LY(s) + (sI - F)^{-1}[\overline{G}_1U(s) + \overline{G}_2Y(s)].$$

(14)

Also, since $\dot{x} = [P \ Q]\dot{w} = Py + Q\dot{w}_2$, we get

$$\dot{X}(s) = PY(s) + Q\dot{W}_2(s).$$

(15)

Since $u = K\dot{x}$, we get

$$U(s) = KPY(s) + KQ\dot{W}_2(s) = KPY(s) + KQLY(s) + KQ(sI - F)^{-1}[\overline{G}_1U(s) + \overline{G}_2Y(s)].$$

(16)

This equation now involves only $Y(s)$ and $U(s)$. Solving for $U(s)$ in terms of $Y(s)$, we get

$$U(s) = \{sI - KQ(sI - F)^{-1}\overline{G}_1\}^{-1}\{KP + KQL + KQ(sI - F)^{-1}\overline{G}_2\}Y(s).$$

(17)

Hence, the transfer function from $y$ to $u$ is given by

$$\{sI - KQ(sI - F)^{-1}\overline{G}_1\}^{-1}\{KP + KQL + KQ(sI - F)^{-1}\overline{G}_2\}.$$  

Another way to obtain the transfer function from $y$ to $u$ is to substitute $u = K[P \ Q]\dot{w} = KPy + KQ\dot{w}_2 = KPy + KQLy + KQz$ into equation (12) to get

$$\dot{z} = Fz + \overline{G}_1(KP + KQLy + KQz) + \overline{G}_2y$$

$$= (F + \overline{G}_1KQ)z + (\overline{G}_1KP + \overline{G}_1KQL + \overline{G}_2)y.$$  

(18)

Hence, the transfer function from $y$ to $z$ is $(sI - F - \overline{G}_1KQ)^{-1}(\overline{G}_1KP + \overline{G}_1KQL + \overline{G}_2)$. Therefore, since $u = KPy + KQLy + KQz$, the transfer function from $y$ to $u$ is given by

$$H_u(s) = KP + KQL + KQ(sI - F - \overline{G}_1KQ)^{-1}(\overline{G}_1KP + \overline{G}_1KQL + \overline{G}_2).$$

Alternatively, the transfer function can be found by writing the observer-controller combination in a state space representation. The combination of the reduced-order observer and the controller can be written using (18) and $u = KPy + KQLy + KQz$ as

$$\dot{z} = A_uz + Bu_y$$

$$u = C_uz + Du_y$$

(19)

(20)

where $A_u = F + \overline{G}_1KQ$, $B_u = \overline{G}_1KP + \overline{G}_1KQL + \overline{G}_2$, $C_u = KQ$, and $D_u = KP + KQL$. Note that the state of this system is $z$, the input is $y$, and the output is $u$. The transfer function from $y$ to $u$ is therefore $H_u(s) = C_u(sI - A_u)^{-1}B_u + D_u$.  

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