Proof of Popov-Belevitch-Hautus (PBH) Tests for Controllability and Observability

Given a linear time-invariant system of the form

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx,
\end{align*}
\]

with \(x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \) and \(C \in \mathbb{R}^{p \times n},\) the PBH test for controllability says that the system (1) is controllable if and only if the rank of the matrix \([sI - A : B]\) is \(n\) for all \(s \in \mathbb{C},\) the set of complex numbers. This rank condition is equivalent to saying that there does not exist a left eigenvector of \(A\) that is orthogonal to columns of \(B.\) The proof of this PBH test has two parts: necessity and sufficiency, as outlined below.

### First part of the proof of PBH test for controllability

i.e., that if the system is controllable, then a left eigenvector of \(A\) that is orthogonal to columns of \(B\) does not exist. To prove this, we proceed by contradiction. If a row vector \(q\) exists which is a left eigenvector of \(A\) and is orthogonal to columns of \(B,\) then we have: \(qA = \lambda q\) and \(qB = 0\) where \(\lambda\) is a scalar constant (the eigenvalue corresponding to the left eigenvector \(q\)). Then, \(q[B, AB, A^2B, \ldots, A^{n-1}B] = 0\) since \(qB\) is zero by definition of \(q;\) \(qAB = \lambda qB = 0;\) \(qA^2B = \lambda qAB = \lambda^2 qB = 0;\) etc. Hence, \([B, AB, A^2B, \ldots, A^{n-1}B]\) has rank less than \(n;\) therefore, the system is not controllable. Hence, we have proven that if a left eigenvector of \(A\) that is orthogonal to columns of \(B\) exists, then the system is not controllable. Therefore, if the system is controllable, then a left eigenvector of \(A\) that is orthogonal to columns of \(B\) does not exist.

### Second part of the proof of PBH test for controllability

i.e., that if a left eigenvector of \(A\) that is orthogonal to columns of \(B\) does not exist, then the system is controllable. To prove this, we again proceed by contradiction. If the system is not controllable, then after a change of coordinates given by \(z = Tx,\) the system can be put into the form \(\dot{z} = A_z z + B_z u\) with \(A_z\) and \(B_z\) being of the form:

\[
A_z = \begin{bmatrix}
\mathcal{A}_{11} & \mathcal{A}_{12} \\
0 & \mathcal{A}_{22}
\end{bmatrix}; \quad B_z = \begin{bmatrix}
\overline{B}_1 \\
0
\end{bmatrix},
\]

i.e., the state \(z = [z_1, z_2]\) with the first part \(z_1\) of the transformed state \(z\) having the dynamics \(\dot{z}_1 = \mathcal{A}_{11} z_1 + \mathcal{A}_{12} z_2\) and the second part \(z_2\) of the transformed state \(z\) having the dynamics \(\dot{z}_2 = \mathcal{A}_{22} z_2,\) i.e., \(z_2\) is the uncontrollable part of the system written in the new state representation \(z.\) Define \(w\) to be a left eigenvector of \(\mathcal{A}_{22},\) i.e., \(w_{22}\) is a row vector such that \(w_{22} \mathcal{A}_{22} = \lambda_{22} w_{22}\) with \(\lambda_{22}\) being a scalar constant.

Define \(w = [0, w_{22}]\) where the 0 in the definition of \(w\) stands for a row vector comprising of all zeros and with the same number of elements as \(z_1.\) Then, we have \(w A_z = [0, w_{22} \mathcal{A}_{22}] = [0, \lambda_{22} w_{22}] = \lambda_{22} w\) and \(w B_z = 0.\) Therefore, \(w\) is a left eigenvector of \(A_z\) that is orthogonal to columns of \(B_z.\) Define \(q = w^T.\) Then, since \(A_z = T A T^{-1}\) and \(B_z = T B,\) we have \(q A = w T T^{-1} A_z T = w A_z T = \lambda_{22} w T = \lambda_{22} q\) and \(q B = w T T^{-1} B_z = w B_z = 0.\) Therefore, \(q\) is a left eigenvector of \(A\) that is orthogonal to columns of \(B.\) Hence, we have proven that if the system is not controllable, then a left eigenvector of \(A\) exists that is orthogonal to columns of \(B.\) Therefore, if a left eigenvector of \(A\) that is orthogonal to columns of \(B\) does not exist, then the system is controllable.

**Proof of the PBH test for observability:** Given the system (1), the PBH test for observability says that the system (1) is observable if and only if the rank of the matrix \(\begin{bmatrix} sI - A & C \end{bmatrix}\) is \(n\) for all \(s \in \mathbb{C}.\) This rank condition is equivalent to saying that there does not exist a right eigenvector of \(A\) that is orthogonal to rows of \(C.\) To prove this PBH test for observability, we can simply use duality and the fact that we proved the PBH test for controllability above. To do this, observe that \(\begin{bmatrix} sI - A & C \end{bmatrix}^T = [sI - A^T : C^T]\) and note that the rank of any matrix is equal to the rank of its transpose. Hence, the PBH test for observability for the pair \((A, C)\) is equivalent to the PBH test for controllability for the pair \((A^T, C^T).\) By duality, we know that the pair \((A^T, C^T)\) is controllable if and only if the pair \((A, C)\) is observable. Since we proved both necessity and sufficiency of the PBH test for controllability above, this duality therefore implies that the PBH test for observability is necessary and sufficient as well.